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SMOLUCHOWSKI COAGULATION EQUATION WITH A SOURCE TERM AND A SINK

by

Claudia Daniela Calin

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Master of Science

in

Applied Mathematics

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UNIVERSITY OF ALBERTA

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Smoluchowski coagulation equation with a source term and a sink** submitted by **Claudia Daniela Calin** in partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics.

ABSTRACT

In this thesis we study the existence and the uniqueness of solutions to the coagulation equation with particle sources and sinks for all time, in suitably defined Banach spaces and for kernels that are bounded independently of the particle sizes. The global existence of solutions is also studied for product-type coagulation kernels.

The failure of the total mass of solutions of the coagulation equation with source terms and kernels of product-type, also known as the gelation phenomenon, is investigated in some particular cases. We also provide an upper and a lower bound on the gelation time.

Based on the method of Laplace transforms, we reveal some formal series solutions of the coagulation equation with constant kernels and time-independent source terms. An asymptotic long-term behaviour is developed for the solution of the coagulation equation in the presence of a source term possessing arbitrary time-dependence.

DEDICATION

To my beautiful mother and my nephew with all my love.

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This thesis would not have been what it is today without the help of a few special people who made a direct contribution, all deserving thanks. First, all my gratitude to my supervisors, Dr. Henry J.J. van Roessel and Dr. Mazi Shirvani, for time and effort spent reading and editing earlier versions of this thesis, for helping to focus my research in the right direction and for encouraging me from the beginning. I would like to thank my external reviewer Dr. Peter Schiavone for the patience in reading my thesis. I would also like to thank Dr. Peter Minev for reading my thesis and chairing my examining committee.

Several chapters of this thesis were based on papers, either published or submitted, and the comments presented there helped me to polish these portions of my thesis. I would also like to thank Claudia Caia, Ovidiu Voitcu and Victor Salamon for giving me helpful advice.

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Chapter 1

Introduction

We divide our introduction into three sections. In the first, we provide some motivation for studying the coagulation equations with particle sources and sinks. In the second, we reiterate some previous results in the mathematical theory of coagulation. In the third, we provide an overview of this thesis.

1.1 Motivation

The science of aerodisperse systems, or aerosols, is the study of solid or liquid particles suspended in a gas. Throughout nature and in many human activities, distributions of aerosols of various types are continually being created, evolve and erode away. Aerosols are created during various natural processes, such as smoke particles from forest fires, sand and dust storms, blowing snow, condensation of water vapour in the atmosphere, volcanic dust, spores from plant life, and radioactive dust in a nuclear reactor accident. Several industrial operations produce aerosols either as an intentional part of the operation or as an undesirable byproduct. Once an aerosol is produced it is usually unstable because it will disappear with time due to evaporation, coagulation, or precipitation. Several mathematical models, which deal with these processes, have been formulated.

An important mechanism in the evolution of disperse systems is the coagulation of particles, observed in various phenomena such as Brownian coagulation, gravitation coagulation (the formation of stars), or polymerization. A physical phenomenon similar to coagulation takes place in physical processes such as the growth of crystals.

Of particular interest to us is the coagulation process of particles in a disperse system governed by the Smoluchowski coagulation equation. The theory of coagulation of aerosols deals with the process of adhesion or coalescence of aerosol particles when they come in contact with one another. The aim of this theory is the description of the particle size distribution as a function of particle size (or volume) and time as the aerosol population (disperse system) undergoes changes due to various physical and chemical transformations. Coagulation forms new particles of volume $x+y$ from the collision of two particles of sizes x and y; the collision rate is proportional to

the number of available particles and to the coagulation kernel, which will be defined below. Emissions increase the number of particles of a specific composition and size, while deposition processes remove particles from the atmosphere.

For a spatially homogeneous aerosol undergoing coagulation with particle sources and sinks, the size distribution density function $u(x,t)$ is governed by the following coagulation equation of Smoluchowski (integro-differential equation):

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy - \int_0^\infty K(x,y)u(x,t)u(y,t)\,dy + S(x,t) - R(t)u(x,t) \tag{1.1}$$

$$u(x,0) = u_0(x) \geq 0 \qquad \text{a.e.} \tag{1.2}$$

where the variables x, y range in $[0,\infty)$, the time variable t ranges in $[0,\infty)$, the function $K(x,y)$ is the coagulation coefficient for particles of sizes (or volumes) x and y, $S(x,t)$ is the rate of addition of fresh particles to the system and $R(t)$ is the rate of removal of particles from the system. Each of the terms will be explained in detail below.

Equation (1.1) models a system of a large number of particles that can coagulate to form larger clusters of particles, with particle sources and sinks. Each particle in the system is assumed to be fully identified by its size (or volume) which is assumed to be a positive real number in the model considered in this thesis. From a physical point of view, the basic mechanisms taken into account are the coalescence of particles to form larger clusters, emissions and depositions (or sources and sinks). It is also assumed that the rate of these reactions depend only on the sizes of the clusters involved in the reaction. Other effects such as: multiple coagulation, fragmentation, condensation, sedimentation and spatial fluctuations are neglected.

The Smoluchowski coagulation equation with source terms and sinks is potentially useful in industrial applications because one might want to exercise some control over the coagulation processes. For instance, it may be desirable to increase or restrict the limiting number of particles of a particular size. One might attempt to achieve this by the introduction of particles of some prescribed size to enable the coagulation process to arrive at some desired limiting state.

The dynamic behaviour of aerosol size distribution is of fundamental interest in atmospheric science, astrophysics and hematology. Coagulation equations arise in a number of problems in the physical sciences, colloid chemistry, aerosol physics, polymer science and astronomy (see e.g. Melzak [33]). Other examples of applications of these models include aggregation of red blood cells and birth-death processes and meteorology. Derivations of similar equations as well as further details and examples, including a 'discrete' summation version of (1.1), can be found in Drake [12] and Dubovskii [13] and references therein. More recent applications of (1.1) can be found in the work of Ernst et al. [18] and Galkin [21].

Interpretation of the terms in equation (1.1): We now briefly discuss the interpretation of each of the terms in the right-hand side of (1.1) (see McLaughlin et al.

[31], Melzak [33], Stewart [46]). Consider a physical system consisting of a sufficiently large number of randomly moving particles, each of which can be described by a non-negative scalar quantity, say mass in some fixed volume of space (in applications, polymer lengths are often considered in place of mass). We assume that the system is homogeneous and unbounded and that interaction occurs only between two particles at a time. It is assumed that the total number of particles is large enough to justify the use of the density function $u(x,t)$ of particles of size x at time t. $u(x,t)\,dx$ represents then the average number of particles having mass in $(x, x+dx)$ at time t. This average and all other averages are referred to a unit volume (say μm^3).

The coagulation kernel $K(x,y)$ models the rate at which particles of mass x coalesce with those of mass y, and is assumed to be known from the physics of the process. The coagulation kernel $K(x,y)$ is introduced by assuming that the average number of coalescences between particles of mass in $(x, x+dx)$ and those having mass in $(y, y+dy)$ is $K(x,y)u(x,t)u(y,t)\,dx\,dy\,dt$ during the time interval $(t, t+dt)$. From a physical point of view, we assume this kernel is symmetric and non-negative, which means $0 \leq K(x,y) = K(y,x)$, for all x and y, since the rate at which a particle of mass x coalesces with a particle of mass y is the same as that for a particle of mass y coalescing with one of mass x.

In the mathematical system given in (1.1), the type of coagulation process which is being considered is contained in the kernel $K(x,y)$. Moreover, in the integro-differential equation formulation of the coagulation equation, the kernel plays the role of the system parameter and varies with the physical processes producing coagulation. The dependence of the kernels $K(x,y)$ on x and y varies according to the particular application of the equations. Examples of kernels include constants, which are used in models where the rate of coagulation is independent of size. Kernels such as $x+y$ reflect a process where the rate of coagulation increases with sizes, while those kernels of product type $(xy)^\alpha, \alpha > 0$, take into account a non-uniform distribution of produced particles. Explicit examples of kernels and applications can be found in [12, 13, 18, 21, 30, 33, 39, 46, 47].

The first two integrals on the right-hand side of (1.1) represent, respectively:

(a) The increase in the number of particles of mass x as the result of the coalescences of particles of mass $x-y$ and mass y $(0 \leq y < x)$ to form a particle of mass x.

(b) The disappearance of particles of mass x, due to their coalescence with particles of mass y $(0 \leq y < \infty)$. The factor $\frac{1}{2}$ takes into account that either a particle of mass $x-y$ coalesces with one of mass y or vice versa.

For an atmospheric aerosol, the source function or the distribution of external sources $S(x,t)$ may represent the rate of introduction of fresh particles from primary sources.

The function $R(t)$ represents removal processes such as particle losses due to participation in the formation of clouds and precipitation.

For physical reasons, the source term $S(x,t)$ and removal term $R(t)$ are both

assumed to be non-negative functions and are assumed to be known. In situations of practical interest, the functional forms of S and R are such that the equation (1.1) must be solved numerically. In addition, we do not assume the source and removal terms to be continuous functions as we would like to "turn them on and off" at various times. Moreover, throughout this thesis, the removal term is assumed to be a locally integrable function on the interval $[0,T)$, for any $T>0$, i.e. $R \in L^1_{loc}[0,T)$.

For a description of physical processes, it is necessary to specify the functional forms for the coagulation coefficient $K(x,y)$, the rate of production of new particles $S(x,t)$ and the rate of particle removal $R(t)$. In a realistic environment, these functions would be nonlinear functions of size (or volume) and time, and would necessitate the numerical solution of equation (1.1) in nearly all cases. It is of interest, however, to examine certain special cases of K, S and R that retain the essential properties of the functions in the more general case.

Another important consideration in solving the Smoluchowski coagulation equation is a realistic choice of the initial size distribution u_0. From a physical point of view, the initial distribution u_0 is assumed to be non-negative, since particles must have non-negative size. The initial size distribution $u(x,0)$ is defined such that $u(x,0)\,dx$ is the total number of particles whose volume lie between x and $x+dx$ per unit volume of air.

The main purpose of this thesis is to study the dynamic behaviour of aerosol size distributions under the influence of the above physical processes.

1.2 Previous work

During the past few years, increasing attention and effort have been given to the mathematical theory of the coagulation equations. In papers dealing with the Smoluchowski's equation without particle source terms and sinks, the main theoretical questions are related to the study of existence and uniqueness of solutions in suitably defined Banach spaces. The existence, uniqueness, and the behaviour of the solutions of the coagulation equation with sources and sinks (1.1) depend on the type of coagulation kernel and coefficient rates assumed in the equation.

Existence and uniqueness of solutions to the coagulation equations with fragmentation (the initial distribution u_0 and therefore the function u possibly enjoying additional regularity properties) have been the subject of study of several papers since the pioneering works of Melzak [33], Aizenman and Bak [1]. For bounded kernels K, global existence and uniqueness of solutions to the coagulation-fragmentation equations (without sources and sinks) is investigated in [1, 31, 33, 3, 41, 40], while the case of unbounded kernels has been studied in [13, 46, 21, 15, 3, 23, 45, 43, 44, 50, 20], assuming certain growth conditions on the coagulation kernel K:

$$K(x,y) \leq C(1+x+y), \forall (x,y) \in \mathbb{R}^2_+, \text{ and for any } C>0. \tag{1.3}$$

For unbounded kernels satisfying condition (1.3), global existence and uniqueness results are available in [13, 21]. For a long time it was believed (see, e.g. [12, 28, 32, 38]) that solutions of the pure coagulation equation with the coagulation kernel of product type $K(x,y) = xy$ exist only for small times. Later, Ernst et al. [18] continued to investigate the product kernels and constructed explicit solutions (which are also unique, see also Dubovskii [13], Theorem 4.2) to the pure coagulation equation by means of the Laplace transform. Using the same method, Shirvani and van Roessel [39] have presented, recently, some results on the pure coagulation equation for more general coagulation kernels $K(x,y) = \theta(x)\theta(y)$, where $\theta(x) = \alpha + \beta x, \forall \alpha, \beta \geq 0$. For the general coagulation equations with fragmentation, Stewart [46] proved a general existence theorem under certain hypotheses on the growth of the coagulation and fragmentation kernels. Solutions are shown to exist in the space $X^+ = \{u \in L^1(0,\infty) : \int_0^\infty (1+x)|u(x)|\,dx < \infty\}$ provided that the initial distribution u_0 belongs also to X^+. Later, Laurençot [30] proved the existence of solutions to the coagulation equations with a weak fragmentation for product-type coagulation kernels in the same Banach space X^+ considered by Stewart [46]. However, a stronger notion of solution has been defined, compared to the weaker notion introduced in [46]. The proof of the theorem is performed using the same weak L^1 compactness methods first used by Stewart [46].

In most previous studies on the existence and uniqueness of solutions, very strong decay conditions on the initial data u_0 are required (see, e.g. Galkin [21]). Moreover, the solutions of the coagulation equations with fragmentation are proved to exist in a complicated Banach setting.

As pointed out earlier, almost all prior work on Smoluchowski's equation has been established either for the case of pure coagulation or coagulation with fragmentation. Little work has been done on the Smoluchowski's equation with source and removal terms (1.1) with regard to the existence problem of equation (1.1), (1.2) in suitably, less complicated Banach spaces with the coagulation kernels of product-type:

$$K(x,y) = (\alpha + \beta x)(\alpha + \beta y), \forall (x,y) \in \mathbb{R}_+^2, \tag{1.4}$$

where α, β are non-negative real numbers. This kernel includes both constant and product kernels as special cases and represents, in particular, the main object of study in this thesis.

The existence of mass-conserving solutions to the pure coagulation equation and the occurence of gelation have been the subject of perennial study in recent years. A physically relevant and mathematically challenging question was to figure out whether the total mass, defined by $M_1(t) = \int_0^\infty xu(x,t)\,dx$ of solutions to the coagulation equation remains constant throughout time evolution. Several works in the physical literature have considered this question for the pure coagulation equation. Either formal arguments or explicit solutions have been provided to show that the conservation of mass holds true for all time $t \in \mathbb{R}_+$ in the following cases:
(A) If $K(x,y) \leq M$, for some constant $M > 0$ and for all $(x,y) \in \mathbb{R}_+^2$ (see [31, 3]).

(B) If $K(x,y) \leq M(x+y+1)$, for some constant $M > 0$ and for all $(x,y) \in \mathbb{R}_+^2$ (see [47, 3]).
(C) If $K(x,y) = (xy)^\alpha$, when $\alpha \in [0, 1/2]$ and for all $(x,y) \in \mathbb{R}_+^2$, (see, e.g. [11, 3, 20]).

For some rate coefficients $K(x,y)$, increasing sufficiently fast with x and y, it has been proved rigorously that $M_1(t)$ cannot be constant for all t and the property $M_1(t) = M_1(0)$, for all $t \in \mathbb{R}_+$ fails to be true. Therefore, there must exist a non-negative time T_{gel} defined by $T_{gel} := \inf\left\{t \in \mathbb{R}_+ : M_1(t) < M_1(0)\right\} \in [0, +\infty]$ such that $M_1(t) = M_1(0)$, for $t \leq T_{gel}$, but $M_1(t) < M_1(0)$, for $t > T_{gel}$ (see [18, 19, 30, 39, 11, 20, 8]). In the literature this phenomenon is known as the gelation problem and it is interpreted physically as corresponding to the occurence of a dynamic phase transition in the system or by the appearance of an infinite "gel" or "superparticle".

This dynamic phase transition has attracted a great deal of interest in recent years, much effort being spent on trying to characterize the coagulation kernels and initial conditions for which gelation occurs as well as the properties of the gelling solutions. Still mathematical proofs of the occurence of gelation including larger classes of coagulation rates and also fragmentation have only been supplied recently (see e.g. Laurençot [30], Escobedo et al. [19], F.P. da Costa [11]).

Of particular interest in studying the coagulation equation is to obtain analytical solutions. The integro-differential coagulation equation is difficult to solve accurately, due to the fact that the limit of integration of the first term depends on the variable x and the integrands are quadratic (the first term, for instance, is of nonlinear Volterra type in the terminology of integral equations). Analytical solutions are available for certain special cases of equation (1.1). Scott [38] obtained solutions to the pure coagulation form of equation (1.1). Drake [12] provided a general survey of the mathematical aspects of the pure coagulation equation. Klett [28] studied steady-state self-preserving solutions in the case of source-enhanced coagulation. Dubovskii [13, 14] proved the convergence of the coagulation with a source to an equilibrium state.

1.3 Overview of this Thesis

In this thesis we conduct a theoretical analysis in the field of the coagulation equations with particle sources and sinks. The thesis is divided into eight chapters. In this section we will provide a summary of each of the subsequent chapters in the thesis.

Chapter 2 contains the basic background needed for a good understanding of the whole thesis. The main purpose of Chapter 2 is to give a survey of the basic concepts and results in nonlinear functional analysis and ordinary differential equations that we use and require frequently in the chapters that follow. Since almost all the results herein are by now classical, no proofs are given. Instead, references to several textbooks and monographs, where the corresponding proofs can be found, are provided.

In Chapter 3, using the technique of re-scaling the time variable t, the general Smoluchowski coagulation equation (1.1) is simplified to a coagulation equation with source terms only.

In Chapter 4, we prove the existence and the uniqueness of the solutions to (1.1), for all time, in suitably defined Banach spaces X, for kernels that are bounded independently of the particle sizes x and y. Throughout Chapter 4, we assume that the initial distribution belongs to the space X^+, while the source term g is assumed to be an essentially bounded and integrable function from the interval $[0, T]$, for any $T > 0$, to the Banach space X. A contraction mapping argument is used in order to prove the existence and uniqueness of a local solution. Moreover, it was proved that the local existence cover the whole interval $t \in [0, \infty)$ and the global existence and uniqueness is valid on the Banach space X.

In Chapter 5, we prove the existence of solutions to (1.1) in a general Banach space

$$X = \left\{ f \in L^1(0, \infty) : \int_0^\infty (1 + \frac{\beta}{\alpha} x) |f(x)| \, dx < \infty \right\}$$

with $\alpha > 0, \beta \geq 0$ positive constants. In the proof of the main theorem (Theorem 5.1) we make use of the same method provided in [30] and [46], investigating the properties of the solutions to the approximating equations. The main goal is to show the weak compactness in $L^1(0, +\infty)$ of the sequence of the approximating solutions.

In Chapter 6, based on the previous work (see, e.g. [18, 30, 39]), we study the existence of mass-conserving solutions to (1.1) and the occurrence of gelation for the class of coagulation kernels (1.4). This chapter is divided into three sections.

In Section 6.1 we recall the notions of mass-conserving solutions and gelation for the pure coagulation equations (without fragmentation, sources and other processes). In this section we give a quick survey of some previous known results on the gelation phenomenon and mass-conserving solutions. We also analyze the possibility of occurence of gelation for some type of coagulation kernels. The main result presented in this section (Theorem 6.1) is a consequence of Laurençot's results and is related to the long-term behaviour of the total mass of solutions u at time t for the coagulation kernels defined in (1.4). As we can see from Theorem 6.1, the total mass depends strongly on the number of clusters of very small size ($x \approx 0$) in the initial distribution and the smaller this amount is, the faster is the decay of the total mass. Moreover, this theorem will provide us with an upper bound on the gelation time T_{gel} which later can be used for computational purposes.

In Section 6.2, we extend the results obtained in Section 6.1 to the coagulation equation with sources. Moreover, we also obtain an upper bound on the gelation time of (1.1). Our main result, Theorem 6.5), proves that the temporal decay of the total mass depends strongly on both the number of clusters of very small size ($x \approx 0$) in the initial distribution u_0 and the number of new particles of very small size in the source term and we deduce that the smaller these quantities are, the faster is the decay of the total mass.

In Section 6.3, based on the method employed by Shirvani and van Roessel [39], for the pure coagulation equation with kernels of the form (1.4) and using Sturms First Comparison Theorem, we obtain a lower bound for the old gelation time (Theorem 6.8). Along with Theorem 6.1, Theorem 6.8 provides us with an upper and a lower bound for the gelation time in the pure coagulation equations.

In Chapter 7, based on the method of Laplace transforms, our main purpose is to reveal some formal series solutions of the coagulation equation with constant kernels. Examples that we provide include both time-dependent and time-independent cases of source terms. Based on a result given by Simons [42], for the time-dependent source terms, we show that the long-term behaviour of the distribution tends to be independent of the initial value and entirely determined by the source term. In the method that we apply, there are no convergence arguments. Questions regarding the convergence have to be answered. For practical applications, either an analytical proof of convergence, or a numerical method of computing the inverse Laplace transform has to be used.

We conclude our thesis in Chapter 8.

Chapter 2

Background

Preliminaries. Notations and Definitions

In this chapter we introduce the basic concepts and results in functional analysis and ordinary differential equations that we require throughout this thesis. Since all the results are classical, whenever necessary, we refer to the appropriate paper, or book, where the relevant details can be found.

Throughout the thesis we use the following notation: $\mathbb{R}_+ = [0, \infty)$; $R_+^* = (0, \infty)$; $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$; $\mathbb{N} = \{0, 1, 2, \ldots\}$; $\mathbb{N}^* = \{1, 2, \ldots, \}$.

(A) An important notion that will be used throughout this thesis is that of an essentially bounded function.

Definition 2.1. (Essentially bounded function) (see, e.g. Bachman et al. [48] p.117, Dunford [16], p.100) A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called essentially bounded if there exists a constant $M \geq 0$ such that $\{x \in \mathbb{R}_+ : |f(x)| > M\}$ is a μ-zero set. The class of all measurable functions defined on $\mathbb{R}_+$ which are essentially bounded on $\mathbb{R}_+$ is denoted by $L^\infty(\mathbb{R}_+)$, i.e.

$$\begin{aligned} L^\infty(\mathbb{R}_+) &:= \{f : \|f\|_{L^\infty} < \infty\} \quad \text{where,} \\ \|f\|_{L^\infty} &:= \text{ess sup}|f(x)| = \inf\{M : |f(x)| \leq M \text{ a.e. on } \mathbb{R}_+\} \\ &= \inf\{M : \mu(\{x \in \mathbb{R}_+ : |f(x)| > M\}) = 0\} \end{aligned} \tag{2.1}$$

It is routine to verify that $L^\infty(\mathbb{R}_+)$, equipped with the norm $\|\cdot\|_{L^\infty}$, is a Banach space (complete normed real linear space) (see, e.g. Barra [4], p.120-151).

(B) Another important notion that we use in Chapter 3 is that of a locally integrable function.

Definition 2.2. (Locally integrable function) Suppose that U is an open set in $\mathbb{R}$ and $f : U \to \mathbb{R}$ is a Lebesgue integrable function. If the Lebesgue integral $\int_K |f(x)|\, dx$ is finite for all compact subsets K in U then f is locally integrable on U. The set of all such functions is denoted by $L^1_{loc}(U)$.

It is worth mentioning here the following inclusion $L^1(U) \subset L^1_{loc}(U)$.

(C) In order to study the existence and uniqueness of solutions to the Smoluchowski coagulation equation (1.1) subject to the initial condition (1.2), we introduce a suitable Banach space. Let φ be a real-valued and measurable function on $\mathbb{R}_+$ satisfying the following hypotheses:

$$\varphi : \mathbb{R}_+ \to R_+^* \text{ such that } \frac{1}{\varphi} \in L^\infty(\mathbb{R}_+). \tag{2.2}$$

$$\varphi \text{ is a subadditive function, i.e. } \varphi(x+y) \leq \varphi(x) + \varphi(y),\ \forall x, y \in \mathbb{R}_+. \tag{2.3}$$

For $f \in L^1(0,\infty)$, define $\|\cdot\|_X$ by $\|f\|_X = \int_0^\infty \varphi(x)|f(x)|\,dx$. It is trivial to verify that $\|\cdot\|_X$ is a norm (possibly infinite-valued). Now define

$$X := \{f \in L^1(0,\infty) : \|f\|_X < \infty\}. \tag{2.4}$$

It is routine to verify that X is indeed a Banach space. It is denoted by

$$X = L^1(0,\infty;\, \varphi(x)\,dx).$$

Let X^+ be the positive cone of the Banach space X, i.e.

$$X^+ = \{f \in X : f \geq 0 \text{ a.e.}\} \tag{2.5}$$

(D) Let T be arbitrary such that $0 < T \leq \infty$ and let us denote by

$$L^1([0,T];X) = \left\{h : [0,T] \to X :\ h \text{ is measurable and } \int_0^T h(t)\,dt < \infty\right\} \tag{2.6}$$

the space of integrable functions from $[0,T]$ into the Banach space X defined above. More details about the integrability of X-valued functions can be found in Dunford [16], p.119, Edwards [17], p.188, Barra [4], p.120-151.

In order to prove the existence of non-negative solutions to the equation (4.17) on some interval $[0,\tau_0]$, for τ_0 sufficiently small, we make use of the Contraction Mapping Theorem. First let us define here the notion of "contraction mapping":

Definition 2.3. (Contraction mapping)(see Mohana [36], p.47, Burton [7], p.72) If $\mathcal{S}$ is a subset of a Banach space B and A is a mapping taking $\mathcal{S}$ into B, (written as $A : \mathcal{S} \to B$), then A is said to be a contraction on $\mathcal{S}$ if there exists a real number α $(0 < \alpha < 1)$ called the contraction constant for A on $\mathcal{S}$ such that:

$$\|Ax - Ay\| \leq \alpha\|x - y\|, \text{ for all } x, y \in \mathcal{S}$$

A fixed point of a transformation $A : B \to B$ is a point $x \in B$ such that $Ax = x$.

Theorem 2.1. (Contraction Mapping Theorem) (see Burton [7], p.72, Mohana [36], p.47) Let $\mathcal{S}$ be a closed subset of a Banach space B and let $A : \mathcal{S} \to \mathcal{S}$ a contraction mapping. Then there exists a unique point $\phi \in \mathcal{S}$ with $A(\phi) = \phi$. In other words, any contraction mapping A of $\mathcal{S}$ into itself has a unique fixed point. Furthermore, if $\psi_0 \in \mathcal{S}$ is an arbitrary element of $\mathcal{S}$, and if $\{\psi_n\}_{n\geq 1}$ is defined inductively by:

$$\psi_1 = A(\psi_0) \text{ and } \psi_{n+1} = A(\psi_n)$$

then $\psi_n \to \phi$, the unique fixed point. □

In order to prove the global uniqueness of solutions, in Chapter 4, we apply Grönwall's inequality.

Lemma 2.1 (Grönwall's inequality). (see Mohana [36], p.39) If, for $t_0 \leq t \leq t_1$, $\phi(t)$ and $\psi(t)$ are continuous, non-negative functions satisfying the inequality:

$$\phi(t) \leq K + C \int_{t_0}^{t} \psi(s)\phi(s)\, ds \text{ for } t_0 \leq t \leq t_1,$$

with K and C non-negative constants, then the inequality

$$\phi(t) \leq K e^{C\int_{t_0}^{t} \psi(s)\, ds} \text{ holds on } t_0 \leq t \leq t_1$$

□

(E) We denote by

$$Y_T := C([0,T]; L^1(0,\infty)) \tag{2.7}$$

the space of all continuous functions from $[0,T]$ into $L^1(0,\infty)$ endowed with the usual sup-norm or uniform norm: $\|f\| := \sup_{t\in[0,T]} |f(t)|$, (see, e.g. Edwards [17], p.203).

Let $\left(L^1(0,\infty)\right)^*$ denote the dual (or the conjugate) space of $L^1(0,\infty)$. Also let $L^1(0,\infty)_w$ denote the space $L^1(0,\infty)$ when endowed with the weak topology generated by the continuous linear functionals on $L^1(0,\infty)$. The family of seminorms $\left\{\rho_h : h \in \left(L^1(0,\infty)\right)^*\right\}$ is defined by $\rho_h(x) = |h(x)|$, for all $x \in L^1(0,\infty)$.

It is well-known that $\left(L^1(0,\infty)\right)^* = L^\infty(0,\infty)$. We recall, for convenience, the notion of weakly continuous function.

Definition 2.4. (Weakly continuous function) A function $f : [0,T] \to L^1(0,\infty)$ is said to be weakly continuous at $t \in [0,T]$ if for every $\phi \in \left(L^1(0,\infty)\right)^*$ we have $\phi(f(t))$ continuous at t.

We denote by

$$C([0,T]; L^1(0,\infty)_w) \tag{2.8}$$

the space of all weakly continuous functions from $[0,T]$ into the Banach space $L^1(0,\infty)$. $C([0,T],L^1(0,\infty)_w)$ is a locally convex topological vector space (see [17], p.77) whose topology is defined by the family of seminorms $\{p_h(g) = \sup_{t\in[0,T]} \rho_h(g(t))\}$, for all $g \in C([0,T];L^1(0,\infty))$.

Strong convergence in the Banach space L^1 will be denoted by '$\rightarrow$', whereas weak convergence is denoted by '$\rightharpoonup$'. Full details about these two types of convergence can be found in [16], [17], [4], p.136. For convenience, we state here the definitions of strong and weak convergence in the Banach space $L^1(0,\infty)$.

Definition 2.5. (Strong Convergence) Let $\{f_n\}_{n\geq 1}$ be a sequence in $L^1(0,\infty)$ and $f \in L^1(0,\infty)$. The sequence $\{f_n\}$ is said to converge strongly to f, written $f_n \rightarrow f$, if for each $\varepsilon > 0$ there is an integer $N = N_\varepsilon$ such that $\int_0^\infty |f_n(x) - f(x)|\,dx < \varepsilon$, for any $n \geq N_\varepsilon$.

Definition 2.6. (Weak Convergence) The sequence $\{f_n\}_{n\geq 1} \in L^1(0,\infty)$ is said to converge weakly to $f \in L^1(0,\infty)$, written $f_n \rightharpoonup f$, if $\int_0^\infty f_n(x)\,dx \rightarrow \int_0^\infty f(x)\,dx$, as $n \rightarrow \infty$ in the strong sense. The function f is called the weak limit of the sequence f_n.

Definition 2.7. (Uniform convergence)(see [4], p.10) Let $\{f_n\}_{n\geq 1}$ and f be real-valued functions on a space S. Then f_n converges uniformly to f, if given $\varepsilon > 0$ there exists n_ε such that $\sup\{|f_n(x) - f(x)| : x \in S\} < \varepsilon$, for $n \geq n_\varepsilon$.

Definition 2.8. (Almost uniform convergence)(see [17], p.200, [16], p.145) Let S be an integrable set and $\{f_n\}_{n\geq 1}$ and f be measurable functions defined on S. The sequence f_n converges to f almost uniformly if given $\varepsilon > 0$, there exists a measurable set $\mathcal{X} \subset S$ (which may be assumed compact) such that $\mu(S \setminus \mathcal{X}) \leq \varepsilon$ and $f_n \rightarrow f$ uniformly on $S \setminus \mathcal{X}$.

It is known from results in functional analysis that:

Theorem 2.2. (see Edwards [17], p.289) A sequence $\{f_n\}_{n\geq 1}$ converges in the weak sense to f in $L^1(0,\infty)$ as $n \rightarrow \infty$ if and only if for each $\xi \in L^\infty(0,\infty)$ there holds:

$$\int_0^\infty \xi(x) f_n(x)\,dx \rightarrow \int_0^\infty \xi(x) f(x)\,dx \text{ as } n \rightarrow \infty$$

□

The proof of the global existence of solutions of the coagulation equation with source term and unbounded kernel (5.4) with initial condition (5.5) is carried out by investigating the properties of the solutions of approximating equations (5.16). The main objective is to show the weak compactness in $L^1(0,\infty)$ of the sequence of approximating solutions, an approach first employed by Stewart in [46] and continued recently by Laurençot [30] in the context of continuous coagulation-fragmentation equations. For this purpose, we state here the main definitions and theorems that are required later in this thesis.

Definition 2.9. (Locally bounded function) A function f from a topological space S to a metric space M is called locally bounded if for any $x \in S$, there exists an open set $\mathcal{U} \supset S$ s.t. f is bounded on $\mathcal{U}$.

Definition 2.10. (Equicontinuity (Weakly equicontinuity))(see Vrabie [49], p.7) Let B be a Banach space and $f \in C([a,b];B)$. A subset $\mathcal{P}$ in $C([a,b];B)$ ($C([a,b];B)$) is called equicontinuous (weakly equicontinuous) at $t_0 \in [a,b]$ if given a neighbourhood (weak neighbourhood) V of the origin in B there exists $\delta(V) > 0$ such that:

$$f(t) - f(t_0) \in V$$

for each $t \in [a,b]$, $|t - t_0| \leq \delta(V)$ and uniformly for $f \in \mathcal{P}$.

The subset $\mathcal{P}$ is called equicontinuous (weakly equicontinuous) on $[a,b]$ if it is equicontinuous (weakly equicontinuous) at each $t \in [a,b]$.

Definition 2.11. (see Edwards [17], p.18, Vrabie [49], p.4) A subset A of a topological space is said to be:

(a) compact, if every generalized sequence in A has at least one generalized subsequence which converges to some element of A;

(b) relatively compact, if its closure $\bar{A}$ is compact;

(c) sequentially compact, if every generalized sequence in A has at least one subsequence which converges to some element of A;

(d) relatively sequentially compact, if its closure $\bar{A}$ is sequentially compact.

(e) weakly relatively compact, if every generalized sequence in A has a subsequence converging weakly to some element of A.

Theorem 2.3. (Arzela-Ascoli theorem)(see Dunford [16], p.266) Let F be a weakly equicontinuous family of functions from $I = [a,b]$ into a Banach space B and let $\{f_n(t)\}$ be a sequence in F such that for each $t \in I$, the set $\{f_n(t) : n \geq 1\}$ is weakly relatively compact. Then there exists a subsequence $\{f_{n_k}(t)\}_{k \geq 1}$ which converges weakly uniformly on I to a weakly continuous function. □

Theorem 2.4. (Variant of Arzela-Ascoli theorem) (see [46]) For each $T \in (0,\infty)$, the sequence $\{f_n\}_{n \geq 1}$ is relatively sequentially compact in $C([0,T];L^1(0,\infty)_w)$ if and only if the sequence $\{f_n\}_{n \geq 1}$ satisfies the following two properties:
(P1) The set $\{f_n(t) : n \geq 1\}$ is weakly relatively compact in $L^1(0,\infty)$ for every $t \in [0,T]$.
(P2) The set $\{f_n(t) : n \geq 1\}$ is weakly equicontinuous in $L^1(0,\infty)$ at every $t \in [0,T]$.
□

In order to verify the first property $(P1)$, we make use of the following theorem:

Theorem 2.5. (Dunford-Pettis theorem) (see Edwards [17], p.274) In order that a subset $\mathcal{P}$ of $L^1(0,\infty)$ be weakly relatively compact, it is necessary and sufficient that the following three conditions be fulfilled:
(P3)

$$\sup\Big\{\int_{\mathbb{R}_+}|f|\,d\mu : f\in\mathcal{P}\Big\}<\infty$$

(P4) given $\varepsilon>0$, there exists a compact set $\mathcal{K}\subset\mathbb{R}_+$ such that

$$\sup\Big\{\int_{\mathbb{R}_+\setminus\mathcal{K}}|f|\,d\mu : f\in\mathcal{P}\Big\}\le\varepsilon$$

(P5) given $\varepsilon>0$, there exists a number $\delta>0$ such that

$$\sup\Big\{\int_{\mathcal{A}}|f|\,d\mu : f\in\mathcal{P}\Big\}\le\varepsilon$$

provided that $\mu(\mathcal{A})\le\delta$, where $\mu(\cdot)$ denotes the Lebesgue measure. □

Lebesgue's Dominated Convergence Theorem and Egoroff's Theorem play an important role when proving the global existence results for the continuous coagulation equation in Chapter 5.

Theorem 2.6. (Lebesgue Dominated Convergence Theorem) (see Barra [4] p.116, Dunford et al. [16], p.151) Let $\{f_n\}_{n\ge1}$ be a sequence of measurable functions in $L^1(0,\infty)$ converging almost everywhere to a function f. Suppose that there exists a function g in $L^1(0,\infty)$ such that $|f_n(s)|\le g(s)$, almost everywhere (a.e). Then f is in $L^1(0,\infty)$ and $\|f_n-f\|_{L^1}:=\int_0^\infty|f_n(s)-f(s)|\,ds\to0$ as $n\to\infty$. □

Theorem 2.7. (Egoroff's Theorem) (see Dunford [16], p.149, Barra [4], p.143) Let $\{f_n\}_{n\ge1}$ and f be measurable functions on a space E so that $f_n\to f$, almost everywhere on E then $f_n\to f$ almost uniformly on E. □

We quote here the following differential inequality, which will prove useful when establishing, in Section 6.3, a lower bound on the gelation time.

Lemma 2.2. (see Hale [24], p.31, [36], p.33) Let $w(t,u)$ be a continuous function on an open connected set $\Omega\subset\mathbb{R}^2$. If u is a solution of $u'(t)=w(t,u(t))$ for $t\in[a,b]$ and v satisfies $v'(t)\ge w(t,v(t))$ on $[a,b)$ with $v(a)\ge u(a)$, then $v(t)\ge u(t)$ for $t\in[a,b]$. □

Since we will be dealing with functions which are the Laplace transforms of positive functions, it is convenient to define the following class of integrable, non-negative functions on some open interval (a,b):

$$L^1_+(a,b):=\{f:(a,b)\to\mathbb{R};\ f\in L^1(a,b),\ f(x)\ge0\}\tag{2.9}$$

The results in Section 6.3 make use of the notion of completely monotonic functions.

Definition 2.12. (Completely monotonic function) (see Widder [51]) A function $f : (a, b) \to \mathbb{R}$ is said to be completely monotonic if $(-1)^k f^{(k)}(x) \geq 0, k = 0, 1, 2, \ldots$ for all x in $[a, b]$.

There is a correspondence between completely monotonic functions and the Laplace transforms of positive functions and this is stated in the following theorem

Theorem 2.8. (see Widder [51]) A function $f(x)$ is completely monotonic for $x \in \mathbb{R}_+$ if and only if it can be represented in the form

$$f(x) = \int_0^\infty e^{-x\lambda} \varphi(\lambda)\, d\lambda,$$

where $\varphi \in L^1_+(0, \infty)$. □

In order to prove the uniqueness theorem in Section 6.3, the locally and globally Lipschitz properties are used. The following theorems give local and global existence and uniqueness of solutions of an initial value problem.
Consider the following initial value problem:

$$\frac{dx}{dt} = f(t, x) \text{ with the initial condition } x(t_0) = x_0,\ t_0 \in \mathbb{R}_+, \tag{2.10}$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$.

Theorem 2.9. (Local existence and uniqueness theorem) (see [36], p.50) The initial value problem (2.10) has a unique solution defined on the interval $[t_0, t_0 + \tau]$ if the function $f(t, x)$ is continuous in a closed rectangular box $D = \{(t, x) : t_0 \leq t \leq t_0 + a,\ \|x - x_0\| \leq b\}$ and satisfies the "local Lipschitz condition":

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|,$$

for some positive constant L and for all $(t, x), (t, y) \in D$. Then there exists $T > t_0$ such that the above IVP has a unique solution on the interval $[t_0, T]$. □

Definition 2.13. (Globally Lipschitz function) A function $f \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ is called globally Lipschitz if there exists a value $L \geq 0$ such that

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|,\ \forall\, t, x, y \in \mathbb{R}_+.$$

These conditions must hold uniformly (i.e. the same Lipschitz constant has to work for all time in the interval $[0, T]$).

Theorem 2.10. (Global existence and uniqueness theorem) (see [36], p.51) Let $f \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ and assume $f(t, x)$ is globally Lipschitz in the sense of Definition 2.13. Then the initial value problem (2.10) has a unique solution $x(t)$ defined for $t \geq t_0$. □

An important theorem which will be used in Section 6.3 is the Sturm's First Comparison Theorem (we will give two variants for it here):

Consider the following linear differential equations having real-valued, continuous coefficient functions $P_i(t), Q_i(t)$, for $i = 1, 2$ throughout the interval $J : [t_0, t^0]$. Moreover, assume that the functions P_i have continuous first derivatives throughout J.

$$L_1(y) : \big(P_1(t)y'(t)\big)' + Q_1(t)y(t) = 0 \tag{2.11}$$

$$L_2(y) : \big(P_2(t)y'(t)\big)' + Q_2(t)y(t) = 0 \tag{2.12}$$

Definition 2.14. (Sturm majorant) (see Hartman [27], p.334) Equation (2.12) is said to be a Sturm majorant for (2.11) on J if the following inequalities hold:

$$P_1(t) \geq P_2(t) > 0 \text{ and } Q_1(t) \leq Q_2(t), \forall t \in J$$

Theorem 2.11. (see, e.g. Coppel [10], Theorem 9, p.14, [6]) Let $y_1(t)$ and $y_2(t)$ be solutions of the equations (2.11), (2.12) respectively. Assume that the coefficient functions P_i, Q_i, for $i = 1, 2$ are continuous on the interval J, and that (2.12) is a Sturm majorant for (2.11) in the sense of Definition 2.14. If t_n', t_n'' are the $n-th$ zeros of $y_1(t), y_2(t)$, then $t_n'' \leq t_n'$. Moreover $t_n'' < t_n'$ if either $P_1(t) > P_2(t)$ and $Q_1^2(t) + Q_2^2(t) > 0$, or $Q_2(t) > Q_1(t)$ holds for at least one point of the interval $[t_0, t_n']$. If either $y_1(t_0) = 0$ or $y_1(t_0) \neq 0, y_2(t_0) \neq 0$ and

$$\frac{P_1(t_0)y_1'(t_0)}{y_1(t_0)} \geq \frac{P_2(t_0)y_2'(t_0)}{y_2(t_0)} \tag{2.13}$$

then $y_2(t)$ has at least as many zeros as $y_1(t)$ in the interval J. □

Another variant of the Sturm's First Comparison Theorem is the following:

Theorem 2.12. (see Hartman [27], p.334) Let the coefficient functions P_i, Q_i in (2.11) with $i = 1, 2$ be continuous on the interval J and let (2.12) be a Sturm majorant for (2.11). Let $y_1(t) \neq 0$ be a solution of (2.11) and $y_1(t)$ has exactly $n\,(n \geq 1)$ zeros $t = t_1 < t_2 < \ldots < t_n$ on $t_0 < t < t^0$. Let $y_2(t) \neq 0$ be a solution of (2.12) satisfying (2.13) at $t = t_0$. Then $y_2(t)$ has at least n zeros on $t_0 < t \leq t_n$. □

Chapter 3

Coagulation equations with sources and sinks

For a spatially homogeneous aerosol undergoing coagulation with particle sources and sinks, the size distribution density function $u(x,t)$ is governed by the following Smoluchowski coagulation equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy - u(x,t)\int_0^\infty K(x,y)u(y,t)\,dy + S(x,t) - R(t)u(x,t), \text{ a.e. } (x \geq 0,\ t \geq 0) \tag{3.1}$$

subject to the initial condition:

$$u(x,0) = u_0(x), \qquad \text{a.e.} \tag{3.2}$$

where the size variables x and y range in $[0,\infty)$, the time variable t ranges in $[0,\infty)$ and $u(x,t)$ is the distribution function of particles with mass x at time t. As was pointed out in Chapter 1, Section 1.1 the reaction rate $K(x,y)$ is a non-negative function, called the coagulation kernel, $S(x,t)$ describes the rate of addition of fresh particles to the system, while the function $R(t)$ describes the rate of removal of particles from the system.

Equations (3.1) and (3.2) describe the time evolution of particles in disperse systems with distribution function $u(x,t)$ of size (or mass) $x \geq 0$ at time $t \geq 0$, whose change in mass is governed by the non-negative reaction kernel $K(x,y)$. Equation (3.1) models a system of a large numbers of particles which coagulate to form larger clusters of particles. The evolution of coagulation systems is mainly determined by this kernel. The dependence of the kernel $K(x,y)$ on the variables x and y varies according to the particular application of the coagulation equations.

Examples of kernels include constant kernels, which are used in models where the rate of coagulation is independent of size. Kernels such as $K(x,y) = \theta(x)\theta(y)$, where $\theta(x) = \alpha + \beta x$, where $\alpha, \beta \geq 0$ are constants, take into account a non-uniform distribution of produced particles. By physical considerations, the coagulation kernel K is assumed to be a non-negative and symmetric function.

The function $S(x,t)$ is the distribution function of external sources, describing the sources of the new particles of mass x at time t. The function $S(x,t)$ may represent the rate of introduction of fresh particles from primary sources or the rate of formation of new particles by homomolecular or heteromolecular nucleation. The function $R(t)$ represents removal processes such as sedimentation or particle loss due to participation in the formation of clouds and precipitation. The functions $S(x,t)$ and $R(t)$ are supposed to be known.

For physical reasons, the source and removal terms are assumed to be non-negative, i.e. $S(x,t) \geq 0$, and $R(t) \geq 0$ for any $x,t \geq 0$. In addition, we do not assume that they are continuous functions, since we would like to allow the source and removal terms to be "turned on" or "turned off" at various times. The following basic assumption on the removal term will be made throughout the thesis:

$$R \in L^1_{loc}[0,T) \qquad \text{where} \quad 0 < T \leq \infty$$

which means R is a locally integrable function on the interval $[0,T)$ (in the sense of Definition 2.2 in Chapter 2). This implies that $\int_0^t R(s)\,ds < \infty$, for any $t \in [0,T)$.

The main objective of the present chapter is to obtain, using a technique of re-scaling the time variable t, a new but simplified form of Smoluchowski coagulation equation (3.1) with source terms only. In order to accomplish this we start by re-writing the equation (3.1) in the equivalent form:

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) + R(t)u(x,t) = \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy \\ - u(x,t)\int_0^\infty K(x,y)u(y,t)\,dy + S(x,t) \qquad \text{a.e. } (x \geq 0,\ t \geq 0)\end{aligned} \tag{3.3}$$

Let $H(t) := e^{\int_0^t R(s)\,ds}$. Multiplying the equation (3.3) by $H(t)$ defined above, equation (3.3) becomes

$$\begin{aligned}\frac{\partial}{\partial t}\big(u(x,t)H(t)\big) = \frac{H(t)}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy \\ - H(t)u(x,t)\int_0^\infty K(x,y)u(y,t)\,dy + H(t)S(x,t)\end{aligned}$$

Put

$$w(x,t) := u(x,t)H(t).$$

Then the equation obtained above becomes:

$$\begin{aligned}\frac{\partial w}{\partial t}(x,t) = \frac{1}{2H(t)}\int_0^x K(x-y,y)w(x-y,t)w(y,t)\,dy \\ - \frac{1}{H(t)}\int_0^\infty K(x,y)w(x,t)w(y,t)\,dy + H(t)S(x,t),\end{aligned} \tag{3.4}$$

subject to the initial condition

$$w(x,0) = u(x,0)H(0) = u(x,0) = u_0(x) \qquad (\text{since } H(0) = 1).$$

Let us now multiply equation (3.4) by $H(t)$. Then we obtain:

$$H(t)\frac{\partial w}{\partial t}(x,t) = \frac{1}{2}\int_0^x K(x-y,y)w(x-y,t)w(y,t)\,dy - \int_0^\infty K(x,y)w(x,t)w(y,t)\,dy + [H(t)]^2 S(x,t) \tag{3.5}$$

Let us re-scale the time variable t. To this purpose, we introduce a new parameter $\tau = F(t)$, where the function $F(t)$ is chosen such that it satisfies the following initial value problem:

$$F'(t) = \frac{1}{H(t)} \text{ with initial condition } F(0) = 0,$$

and define:

$$c(x,\tau) := w(x,t) \tag{3.6}$$

Since $\tau = F(t)$, we have $t = F^{-1}(\tau)$. Moreover, $\dfrac{dt}{d\tau} = \dfrac{1}{F'(t)} = H(t)$. Therefore,

$$\frac{\partial c}{\partial \tau}(x,\tau) = H(t)\frac{\partial w}{\partial t}(x,t) \tag{3.7}$$

Substituting (3.6), (3.7) into the equation (3.5) and denoting by:

$$g(x,\tau) := [H(F^{-1}(\tau))]^2 S(x, F^{-1}(\tau)) \geq 0, \tag{3.8}$$

the equation (3.5) becomes:

$$\frac{\partial c}{\partial \tau}(x,\tau) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,\tau)c(y,\tau)\,dy - \int_0^\infty K(x,y)c(x,\tau)c(y,\tau)\,dy + g(x,\tau) \tag{3.9}$$

subject to initial condition

$$c(x,\tau=0) = w(x,t=0) = u(x,0) \qquad \text{a.e., or}$$
$$c(x,0) = u_0(x) =: c_0(x) \tag{3.10}$$

Remark. As a consequence of time variable re-scaling, the general Smoluchowski coagulation equation (3.1), (3.2) with source and sink terms simplifies to the Smoluchowski coagulation equation (3.9), (3.10) with a source term only.

Chapter 4

Bounded kernels: Existence and uniqueness of solutions

In this chapter we are interested in studying the global existence and uniqueness of solutions of the following Smoluchowski coagulation equation with source term $g(x,\tau)$:

$$\frac{\partial c}{\partial \tau}(x,\tau) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,\tau)c(y,\tau)\,dy - \int_0^\infty K(x,y)c(x,\tau)c(y,\tau)\,dy + g(x,\tau), \qquad \text{a.e.} \tag{4.1}$$

subject to initial condition

$$c(x,0) = c_0(x) \geq 0 \qquad \text{a.e.} \tag{4.2}$$

Throughout this chapter, the coagulation kernel K is assumed to satisfy the following hypotheses:

$$K(x,y) \geq 0 \text{ is a non-negative, continuous function } \forall x,y \geq 0 \tag{4.3}$$

$$K(x,y) = K(y,x), \ \forall x,y \geq 0, \text{ i.e. } K \text{ is symmetric.} \tag{4.4}$$

$$K(x,y) \leq M, \ \forall x,y \geq 0, \text{ for some constant } M > 0. \tag{4.5}$$

Let φ be a real-valued and measurable function on $\mathbb{R}_+$ satisfying the hypotheses (2.2) and (2.3) in Chapter 2. It follows from (2.2) that there exists a constant $B \in (0,\infty)$ such that:

$$\|1/\varphi\|_{L^\infty} \leq B \ \text{ a.e.} \tag{4.6}$$

Let $T \in (0,\infty]$ be arbitrary. We assume that the initial distribution $c_0 \in X^+$ i.e. it satisfies:

$$\|c_0\|_X = \int_0^\infty \varphi(x)c_0(x)dx < \infty, \tag{4.7}$$

where X^+ is the positive cone of the Banach space X as defined in (2.5) and (2.4), respectively. We also assume that the distribution function $g(x,\tau)$ of the external sources satisfies the following hypotheses:
(A) $g \in L^1([0,T];X)$; and
(B) For any $T > 0$, ess $\sup_{0\le\tau\le T} \int_0^\infty \varphi(x)g(x,\tau)\,dx =$ ess $\sup_{0\le\tau\le T}\|g(\tau)\|_X = C_1(T)$ is finite and positive. (with ess sup defined in (2.1), Chapter 2).

We investigate the existence and uniqueness of solutions to (4.1) and (4.2) for the class of kernels K, the distribution function g and the initial data c_0 described above. Before stating our results, let us define the notion of solution to the equation (4.1) which we will use further.

Definition 4.1. Let $T \in (0,\infty]$ be arbitrary. A solution c of the equation (4.1) is a function $c : [0,T) \to X^+$ such that, for every $\tau \in [0,T)$, there holds

(a) $c \in C([0,T];L^1(0,\infty)) \cap L^\infty(0,\tau;X)$.

(b)

$$(x,y,s) \mapsto K(x,y)c(x,s)c(y,s) \in L^1\big((0,\infty)\times(0,\infty)\times(0,T)\big) \tag{4.8}$$

(c) For almost every $x \in \mathbb{R}_+$:

$$c(x,\tau) = c_0(x) + \int_0^\tau g(x,s)ds + \frac{1}{2}\int_0^\tau\int_0^x K(x-y,y)c(x-y,s)c(y,s)\,dy\,ds - \int_0^\tau c(x,s)\int_0^\infty K(x,y)c(y,s)\,dy\,ds. \tag{4.9}$$

with c_0 and g satisfying (4.7) and (A), (B), respectively.

Existence and uniqueness of global solutions to the coagulation-fragmentation equation with bounded coagulation and fragmentation kernels have been intensively studied since the pioneering work of Melzak [33], continued by Aizenman and Bak [1] and McLaughlin et al. [31]. Our main result in this chapter establishes the existence and uniqueness of global solutions to the coagulation equation with a source term (4.1) subject to the initial condition (4.2).

Theorem 4.1. (Global existence and uniqueness of solutions of equation (4.1)) Let $c_0(x)$ and $g(x,\tau)$ satisfy the hypotheses (4.7) and (A), (B) above. Let φ be a real-valued and measurable function on $\mathbb{R}_+$ satisfying (2.2), (2.3) in Chapter 2 and (4.6). Assume that the coagulation kernel $K(x,y)$ satisfies the hypotheses (4.3), (4.4), (4.5). Then the equation (4.1) subject to the initial condition (4.2) has a unique, non-negative solution $c \in X$ defined on $[0,T)$ (where $T = \infty$) with $c(0) = c_0$. Moreover, for all $\tau \in [0,T)$:

$$\int_0^\infty \varphi(x)c(x,\tau)\,dx \le \int_0^\infty \varphi(x)c_0(x)\,dx + \int_0^\tau\int_0^\infty \varphi(x)g(x,s)\,dx\,ds \tag{4.10}$$

In order to prove the theorem above we introduce first a sequence of useful notations and lemmas. Let $T > 0$ be arbitrary. Equation (4.1) can be rewritten in the equivalent form:

$$\frac{\partial c}{\partial \tau}(x,\tau) + c(x,\tau)\int_0^\infty K(x,y)c(y,\tau)\,dy = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,\tau)c(y,\tau)\,dy + g(x,\tau) \tag{4.11}$$

Consider the Banach space $Y_T := C([0,T]; L^1(0,\infty))$ defined in (2.7), Chapter 2. For all $x \in [0,\infty)$, $\tau \in [0,T]$ we define:

$$N(x,\tau,f) := \frac{1}{2}\int_0^x K(x-y,y)f(x-y,\tau)f(y,\tau)\,dy, \qquad \text{for any } f \in Y_T \tag{4.12}$$

$$\psi(x,\tau,f) := \int_0^\tau \int_0^\infty K(x,y)f(y,s)\,dy\,ds, \qquad \text{for any } f \in Y_T \tag{4.13}$$

Multiplying (4.11) by the integrating factor $e^{\psi(x,\tau,c)}$ we get:

$$\frac{\partial}{\partial \tau}\left(c(x,\tau)e^{\psi(x,\tau,c)}\right) = e^{\psi(x,\tau,c)}\left(g(x,\tau) + N(x,\tau,c)\right) \text{ a.e. } (x \geq 0,\ \tau \geq 0) \tag{4.14}$$

$$c(x,0) = c_0(x) \tag{4.15}$$

For $f \in Y_T$ we define the non-linear coagulation operator G by:

$$G(f)(x,\tau) := c_0(x)e^{-\psi(x,\tau,f)} + \int_0^\tau e^{\psi(x,s,f)-\psi(x,\tau,f)}g(x,s)\,ds + \int_0^\tau e^{\psi(x,s,f)-\psi(x,\tau,f)}N(x,s,f)\,ds \tag{4.16}$$

Integrating (4.11) between 0 and τ we obtain the following integral equation for $c(x,\tau)$

$$c(x,\tau) = G(c)(x,\tau) \text{ for all } x,\tau \geq 0. \tag{4.17}$$

The first step in the proof of the theorem stated above is the proof of the (local) existence and uniqueness of a non-negative solution to the equation (4.17) on some interval $[0,\tau_0]$ for τ_0 sufficiently small chosen below. In order to accomplish this we introduce some more useful definitions and lemmas which will be used to show that the non-linear operator G has a unique fixed point, by means of the Contraction Mapping Theorem (Theorem 2.1, Chapter 2). Fix T, and denote by

$$L := MB \text{ (see (4.5) and (4.6)).}$$
$$R := 2\|c_0\|_X + 2TC_1(T). \tag{4.18}$$

Choose $\tau_0 \in (0,T)$ small enough such that the following inequalities hold:

$$e^{LR\tau}\big(\|c_0\|_X + \tau C_1(T) + R^2 L\tau\big) \le R \qquad \text{for } 0 \le \tau \le \tau_0 \tag{4.19}$$

$$e^{LR\tau} L\tau\big(\|c_0\|_X + \tau C_1(T) + R^2 L\tau + 2R\big) < 1 \qquad \text{for } 0 \le \tau \le \tau_0 \tag{4.20}$$

For τ_0 and R defined, let us endow the Banach space $Y_{\tau_0} := C([0,\tau_0]; L^1(0,\infty))$ with the following norm:

$$\|f\|_1 := \sup_{0\le s\le\tau_0} \int_0^\infty \varphi(x)|f(x,s)|\,ds = \sup_{0\le s\le\tau_0} \|f(s)\|_X \text{ for all } f \in Y_{\tau_0}. \tag{4.21}$$

Let us introduce some additional lemmas to simplify the computation:

Lemma 4.1. For $x \in [0,\infty)$, $0 \le s \le \tau \le \tau_0$, $u \in Y_{\tau_0}$ there holds:

$$e^{|\psi(x,s,u)-\psi(x,\tau,u)|} \le e^{L(\tau-s)\|u\|_1} \tag{4.22}$$

In particular, the following inequality holds

$$e^{|\psi(x,\tau,u)|} \le e^{L\tau\|u\|_1} \tag{4.23}$$

Proof. Let us prove that (4.22) holds. Using definition (4.13) of ψ, the hypothesis (4.5) on K, Hölder's inequality and the property (4.6) of φ we get

$$\begin{aligned}
&\big|\psi(x,s,u) - \psi(x,\tau,u)\big| \\
&= \Big| \int_0^s \int_0^\infty K(x,y)u(y,\sigma)\,dy\,d\sigma - \int_0^\tau \int_0^\infty K(x,y)u(y,\sigma)\,dy\,d\sigma \Big| \\
&= \Big| \int_s^\tau \int_0^\infty K(x,y)u(y,\sigma)\,dy\,d\sigma \Big| \\
&\le \int_s^\tau \int_0^\infty K(x,y)|u(y,\sigma)|\,dy\,d\sigma \\
&\le M \int_s^\tau \int_0^\infty \varphi(y)|u(y,\sigma)|\frac{1}{\varphi(y)}\,dy\,d\sigma \\
&\le MB \int_s^\tau \sup_{0\le\sigma\le\tau\le\tau_0} \|u(\sigma)\|_X\,d\sigma = L(\tau-s)\|u\|_1
\end{aligned} \tag{4.24}$$

Therefore, relation (4.22) holds. In particular, letting $s \equiv 0$ in (4.24), and using definition (4.13) of ψ, we have $\psi(x,0,u) = 0$ and (4.24) implies

$$|\psi(x,\tau,u)| \le L\tau\|u\|_1$$

which proves that relation (4.23) holds. Thus the proof of Lemma 4.1 is now complete. □

For $x \in [0,\infty)$, $0 \le s \le \tau \le \tau_0$ and fixed $u, v \in Y_{\tau_0}$ define:

$$P(x,s,\tau) := e^{\psi(x,s,u)-\psi(x,\tau,u)} - e^{\psi(x,s,v)-\psi(x,\tau,v)} \text{ and denote by}$$

$$C := max\big\{\|u\|_1, \|v\|_1\big\}$$

Lemma 4.2. For $x \in [0,\infty)$, $0 \le s \le \tau \le \tau_0$, $u, v \in Y_{\tau_0}$ there holds

$$|P(x,s,\tau)| \le (\tau - s)L\|u - v\|_1 e^{LC(\tau - s)} \tag{4.25}$$

Proof. We consider two cases.

Case 1 Suppose $e^{\psi(x,s,u)-\psi(x,\tau,u)} \le e^{\psi(x,s,v)-\psi(x,\tau,v)}$ holds for any $u, v \in Y_{\tau_0}$ fixed and $x \ge 0$. Then

$$\begin{aligned} |P(x,s,\tau)| &= -P(x,s,\tau) \\ &= e^{\psi(x,s,v)-\psi(x,\tau,v)}\Big\{1 - e^{\psi(x,s,u)-\psi(x,\tau,u)-[\psi(x,s,v)-\psi(x,\tau,v)]}\Big\} \end{aligned}$$

Using definition (4.13) of ψ, the hypothesis (4.5) on K, Hölder's inequality and the property (4.6) of φ we get

$$\begin{aligned} &\big[\psi(x,s,v) - \psi(x,s,u)\big] - \big[\psi(x,\tau,v) - \psi(x,\tau,u)\big] \\ &= \int_0^\tau \int_0^\infty K(x,y)\big[u(y,\sigma) - v(y,\sigma)\big]\,dy\,d\sigma - \int_0^s \int_0^\infty K(x,y)\big[u(y,\sigma) - v(y,\sigma)\big]\,dy\,d\sigma \\ &= \int_s^\tau \int_0^\infty K(x,y)\big[u(y,\sigma) - v(y,\sigma)\big]\,dy\,d\sigma \\ &\le M\int_s^\tau \int_0^\infty \Big[\big|u(y,\sigma) - v(y,\sigma)\big|\varphi(y)\Big]\frac{1}{\varphi(y)}\,dy\,d\sigma \\ &\le MB(\tau - s)\|u - v\|_1 = L(\tau - s)\|u - v\|_1 \end{aligned}$$

Therefore,

$$\psi(x,s,v) - \psi(x,\tau,v) - [\psi(x,s,u) - \psi(x,\tau,u)] \le L(\tau - s)\|u - v\|_1 \tag{4.26}$$

Take now $u \equiv 0$, then using definition (4.13) of ψ we have, $\psi(x,s,0) = \psi(x,\tau,0) = 0$. From the inequality (4.26) and the definition of C we obtain:

$$\psi(x,s,v) - \psi(x,\tau,v) \le L(\tau - s)\|v\|_1 \le LC(\tau - s). \tag{4.27}$$

Using the inequality: $1 - e^{-y} \le y$ which holds for any $y \ge 0$, we obtain:

$$|P(x,s,\tau)| \le e^{\psi(x,s,v)-\psi(x,\tau,v)}\Big\{\psi(x,s,v) - \psi(x,\tau,v) - [\psi(x,s,u) - \psi(x,\tau,u)]\Big\}$$

Substituting the inequalities (4.26) and (4.27) in the inequality above, relation (4.25) holds.

Case 2 If $e^{\psi(x,s,u)-\psi(x,\tau,u)} \ge e^{\psi(x,s,v)-\psi(x,\tau,v)}$ then inequality (4.25) can be similarly derived. Indeed, we have

$$
\begin{aligned}
|P(x,s,\tau)| &= P(x,s,\tau) \\
&= e^{\psi(x,s,u)-\psi(x,\tau,u)}\Big\{1 - e^{\psi(x,s,v)-\psi(x,\tau,v)-[\psi(x,s,u)-\psi(x,\tau,u)]}\Big\} \\
&\le e^{\psi(x,s,u)-\psi(x,\tau,u)}\Big\{\psi(x,s,u) - \psi(x,\tau,u) - [\psi(x,s,v) - \psi(x,\tau,v)]\Big\}
\end{aligned}
$$

Using now the same arguments as in Case 1, we get:

$$
\begin{aligned}
&\psi(x,s,u) - \psi(x,\tau,u) - \big[\psi(x,s,v) - \psi(x,\tau,v)\big] \\
&= \int_s^\tau \int_0^\infty K(x,y)\big[v(y,\sigma) - u(y,\sigma)\big]\, dy\, d\sigma \\
&\le \int_s^\tau \int_0^\infty K(x,y)|v(y,\sigma) - u(y,\sigma)|\, dy\, d\sigma \le MB(\tau - s)\|u - v\|_1 \\
&= L(\tau - s)\|u - v\|_1
\end{aligned}
$$

Moreover, for $v \equiv 0$ we have $e^{\psi(x,s,u)-\psi(x,\tau,u)} \le e^{LC(\tau-s)}$. Therefore relation (4.25) holds again in Case 2 and the proof of the lemma is completed. □

To obtain the local existence and uniqueness of a solution to the initial value problem (4.1) with initial condition (4.2) we use a contraction mapping argument in the set Y_{R,τ_0} (which depends on R given by (4.18) and τ_0 satisfying (4.19), (4.20)), defined by

$$Y_{R,\tau_0} := \{u \in Y_{\tau_0} : \|u\|_1 \le R\}$$

Let us establish now that for small $\tau_0 > 0$ the closed ball Y_{R,τ_0} in Y_{τ_0} is invariant under the mapping G.

Lemma 4.3. The nonlinear operator G defined by (4.16) maps the set Y_{R,τ_0} into itself, i.e.

$$G(Y_{R,\tau_0}) \subseteq Y_{R,\tau_0}$$

Proof Choose $u \in Y_{R,\tau_0}$, so $\|u\|_1 \le R$. For $\tau \in [0,\tau_0]$, using the definition of G given in (4.16) and Fubini's theorem, we obtain

$$
\begin{aligned}
\|G(u)(\tau)\|_X &= \int_0^\infty \varphi(x)|G(u)(x,\tau)|\, dx \\
&\le \int_0^\infty \varphi(x)c_0(x)e^{-\psi(x,\tau,u)}\, dx + \int_0^\tau \int_0^\infty \varphi(x)e^{\psi(x,s,u)-\psi(x,\tau,u)}g(x,s)\, dx\, ds \\
&+ \int_0^\tau \int_0^\infty \varphi(x)e^{\psi(x,s,u)-\psi(x,\tau,u)}|N(x,s,u)|\, dx\, ds \\
&= (i) + (ii) + (iii)
\end{aligned}
$$

where we have denoted the terms in the right-hand side above by (i), (ii) and (iii), respectively. We have the following estimates for each of them:

$$\begin{aligned}(i) &= \int_0^\infty c_0(x)\varphi(x)e^{-\psi(x,\tau,u)}\,dx \le \int_0^\infty c_0(x)\varphi(x)e^{|\psi(x,\tau,u)|}\,dx \\ &\le e^{LR\tau}\int_0^\infty \varphi(x)c_0(x)\,dx = e^{LR\tau}\|c_0\|_X \qquad \text{(using (4.23))}\end{aligned}$$

$$\begin{aligned}(ii) &= \int_0^\tau\int_0^\infty \varphi(x)g(x,s)e^{\psi(x,s,u)-\psi(x,\tau,u)}\,dx\,ds \\ &\le \int_0^\tau\int_0^\infty \varphi(x)g(x,s)e^{|\psi(x,s,u)-\psi(x,\tau,u)|}\,dx\,ds \\ &\le \int_0^\tau\int_0^\infty e^{LR(\tau-s)}\varphi(x)g(x,s)\,dx\,ds \qquad \text{(using (4.22))} \\ &\le \int_0^\tau e^{LR(\tau-s)}\Big\{\operatorname{ess\,sup}_{0\le s\le\tau_0\le T}\int_0^\infty \varphi(x)g(x,s)\,dx\Big\}\,ds \\ &= C_1(T)\int_0^\tau e^{LR(\tau-s)}\,ds \le C_1(T)\tau e^{LR\tau}\end{aligned}$$

Using definition (4.12) of N, the transformations $x-y=x'$, $y=y'$ and (4.25), the third term (iii) becomes:

$$\begin{aligned}(iii) &= \int_0^\tau\int_0^\infty \varphi(x)e^{\psi(x,s,u)-\psi(x,\tau,u)}\big|N(x,s,u)\big|\,dx\,ds \\ &\le \frac{1}{2}\int_0^\tau\int_0^\infty \varphi(x)e^{|\psi(x,s,u)-\psi(x,\tau,u)|}\int_0^x K(x-y,y)|u(x-y,s)|\cdot|u(y,s)|\,dy\,dx\,ds \\ &\le \frac{M}{2}\int_0^\tau\Big\{\int_0^\infty\int_0^\infty \varphi(x+y)|u(x,s)|\cdot|u(y,s)|\,dy\,dx\Big\}e^{L(\tau-s)R}\,ds\end{aligned}$$

Using (2.3), Hölder's inequality and (4.6) in the above inequality we obtain:

$$\begin{aligned}(iii) &\le \frac{M}{2}\int_0^\tau e^{LR(\tau-s)}\Big\{\int_0^\infty\int_0^\infty [\varphi(x)+\varphi(y)]\cdot|u(x,s)|\cdot|u(y,s)|\,dy\,dx\Big\}\,ds \\ &= M\int_0^\tau e^{LR(\tau-s)}\Big(\int_0^\infty \varphi(y)|u(y,s)|\,dy\Big)\Big(\int_0^\infty \varphi(x)|u(x,s)|\frac{1}{\varphi(x)}\,dx\Big)\,ds \\ &\le MB\|u\|_1^2\int_0^\tau e^{LR\tau}\,ds \le LR^2\tau e^{LR\tau}\end{aligned}$$

Using (i), (ii), (iii) above we get the following estimate of $\|G(u)(\tau)\|_X$

$$\|G(u)(\tau)\|_X \le e^{LR\tau}\big(\|c_0\|_X + \tau C_1(T) + R^2 L\tau\big) \le R \qquad \text{(using (4.19))}$$

If we take $\sup_{0\le\tau\le\tau_0}$ in the above inequality we get:

$$\|G(u)\|_1 \le R.$$

which completes the proof of Lemma 4.3. □

We next show that G is a contraction mapping.

Lemma 4.4. For $u, v \in Y_{R,\tau_0}$ there exists $\delta \in (0,1)$ such that:

$$\|G(u) - G(v)\|_1 \le \delta \|u - v\|_1 \tag{4.28}$$

Proof Let $u, v \in Y_{R,\tau_0}$, so $\|u\|_1 \le R$ and $\|v\|_1 \le R$. Using the definition (4.16) of G we have:

$$\begin{aligned} G(u)(x,\tau) - G(v)(x,\tau) &= c_0(x)\big[e^{-\psi(x,\tau,u)} - e^{-\psi(x,\tau,v)}\big] \\ &+ \int_0^\tau \Big\{e^{\psi(x,s,u)-\psi(x,\tau,u)} - e^{\psi(x,s,v)-\psi(x,\tau,v)}\Big\} g(x,s)\,ds \\ &+ \int_0^\tau \Big\{e^{\psi(x,s,u)-\psi(x,\tau,u)} N(x,s,u) - e^{\psi(x,s,v)-\psi(x,\tau,v)} N(x,s,v)\Big\}\,ds \end{aligned} \tag{4.29}$$

Next, in the last integral, denoted by (III), add and subtract $e^{\psi(x,s,v)-\psi(x,\tau,v)} N(x,s,u)$. Then, using the definition (4.12) of N, the last integral becomes

$$\begin{aligned} (III) &= \int_0^\tau \Big\{e^{\psi(x,s,u)-\psi(x,\tau,u)} - e^{\psi(x,s,v)-\psi(x,\tau,v)}\Big\} N(x,s,u)\,ds \\ &+ \int_0^\tau \Big\{e^{\psi(x,s,v)-\psi(x,\tau,v)} N(x,s,u) - e^{\psi(x,s,v)-\psi(x,\tau,v)} N(x,s,v)\Big\}\,ds \\ &= \int_0^\tau \Big\{e^{\psi(x,s,u)-\psi(x,\tau,u)} - e^{\psi(x,s,v)-\psi(x,\tau,v)}\Big\} N(x,s,u)\,ds \\ &- \int_0^\tau e^{\psi(x,s,v)-\psi(x,\tau,v)}[N(x,s,v) - N(x,s,u)]\,ds \end{aligned} \tag{4.30}$$

According to the definition (4.12) of N if, in the difference $[N(x,s,v) - N(x,s,u)]$ above, we add and subtract $\frac{1}{2}\int_0^x K(x-y,y)v(x-y,s)u(y,s)\,dy$ and we group the terms two by two, the difference in the last integral of (4.30) takes the form:

$$\begin{aligned} N(x,s,v) - N(x,s,u) = \Big\{ &\frac{1}{2}\int_0^x K(x-y,y)v(x-y,s)\big[v(y,s) - u(y,s)\big]\,dy \\ &+ \frac{1}{2}\int_0^x K(x-y,y)u(y,s)[v(x-y,s) - u(x-y,s)]\,dy\Big\}\,ds \end{aligned} \tag{4.31}$$

Substituting (4.31) into (4.30) and (4.30) into (4.29) we obtain:

$$
\begin{aligned}
G(u)(x,\tau) - G(v)(x,\tau) &= c_0(x)\left[e^{-\psi(x,\tau,u)} - e^{-\psi(x,\tau,v)}\right] \\
&\quad + \int_0^\tau \left\{e^{\psi(x,s,u)-\psi(x,\tau,u)} - e^{\psi(x,s,v)-\psi(x,\tau,v)}\right\} g(x,s)\,ds \\
&\quad + \int_0^\tau \left\{e^{\psi(x,s,u)-\psi(x,\tau,u)} - e^{\psi(x,s,v)-\psi(x,\tau,v)}\right\} N(x,s,u)\,ds \\
&\quad - \int_0^\tau \left(e^{\psi(x,s,v)-\psi(x,\tau,v)}\right)\left\{\frac{1}{2}\int_0^x K(x-y,y)\times\right. \\
&\quad \times v(x-y,s)[v(y,s)-u(y,s)]dy \\
&\quad \left. + \frac{1}{2}\int_0^x K(x-y,y)u(y,s)[v(x-y,s)-u(x-y,s)]\,dy\right\} ds
\end{aligned}
$$

Using the definition of $P(x,s,\tau)$ we get

$$
\begin{aligned}
G(u)(x,\tau) - G(v)(x,\tau) &= c_0(x)P(x,0,\tau) + \int_0^\tau P(x,s,\tau)g(x,s)\,ds \\
&\quad + \int_0^\tau P(x,s,\tau)N(x,s,u)\,ds - \int_0^\tau e^{\psi(x,s,v)-\psi(x,\tau,v)} \\
&\quad \times \left\{\frac{1}{2}\int_0^x K(x-y,y)v(x-y,s)\left[v(y,s)-u(y,s)\right]dy\right. \\
&\quad \left. + \frac{1}{2}\int_0^x K(x-y,y)u(y,s)[v(x-y,s)-u(x-y,s)]\,dy\right\} ds \\
&= (1)+(2)+(3)+(4)+(5) \qquad (4.32)
\end{aligned}
$$

Multiplying (4.32) by $\varphi(x)$ and integrating $\varphi(x)|G(u)(x,\tau)-G(v)(x,\tau)|$ with respect to $x \in [0,\infty)$ each of the above terms may be written as follows:

$$
\int_0^\infty \varphi(x)\cdot(1)\,dx \le \int_0^\infty \varphi(x)c_0(x)|P(x,0,\tau)|\,dx \le L\tau e^{LR\tau}\|c_0\|_X\|u-v\|_1 \qquad (4.33)
$$

where we have used (4.25) with $s \equiv 0$.

$$
\begin{aligned}
\int_0^\infty \varphi(x)\cdot(2)\,dx &\le \int_0^\tau \int_0^\infty \varphi(x)g(x,s)\big|P(x,s,\tau)\big|\,dx\,ds \\
&\le \int_0^\tau \int_0^\infty Le^{LR(\tau-s)}(\tau-s)\varphi(x)g(x,s)\|u-v\|_1\,dx\,ds \\
&= L\|u-v\|_1 \int_0^\tau (\tau-s)e^{LR(\tau-s)}\Big\{\int_0^\infty \varphi(x)g(x,s)\,dx\Big\}\,ds \\
&\le L\|u-v\|_1 C_1(T)\int_0^\tau \tau e^{LR\tau}\,ds \le L\tau^2 C_1(T)\|u-v\|_1 e^{LR\tau} \qquad (4.34)
\end{aligned}
$$

$$\int_0^\infty \varphi(x)\cdot(3)\,dx \le \int_0^\tau\int_0^\infty \varphi(x)|P(x,s,\tau)|\cdot|N(x,s,u)|\,dx\,ds$$
$$\le \frac{1}{2}\int_0^\tau\int_0^\infty\int_0^\infty \varphi(x+y)K(x,y)|u(x,s)|\cdot|u(y,s)|\times$$
$$\times(\tau-s)Le^{LR(\tau-s)}\|u-v\|_1\,dy\,dx\,ds$$

Using the subadditivity property (2.3) of φ and the symmetry property (4.4) of the coagulation kernel K we obtain the inequalities

$$\int_0^\infty \varphi(x)\cdot(3)\,dx \le \frac{ML}{2}\int_0^\tau (\tau-s)e^{LR(\tau-s)}\|u-v\|_1\int_0^\infty\int_0^\infty [\varphi(x)+\varphi(y)]\cdot$$
$$\cdot|u(x,s)||u(y,s)|\,dy\,dx\,ds$$
$$\le ML\|u-v\|_1\int_0^\tau\Big\{\int_0^\infty \varphi(x)|u(x,s)|\,dx\Big\}\Big\{\int_0^\infty \varphi(y)|u(y,s)|\frac{1}{\varphi(y)}\,dy\Big\}\times$$
$$\times(\tau-s)e^{LR(\tau-s)}\,ds$$
$$\le MLR^2\|u-v\|_1 B\int_0^\tau(\tau-s)e^{LR(\tau-s)}\,ds$$
$$= L^2R^2\|u-v\|_1\int_0^\tau \tau e^{LR\tau}\,ds \le (LR\tau)^2\|u-v\|_1 e^{LR\tau} \tag{4.35}$$

Using the inequality (4.22) in Lemma 4.1, the fourth term in the equality (4.32) can be written as:

$$\int_0^\infty \varphi(x)\cdot(4)\,dx \le \int_0^\tau\int_0^\infty \varphi(x)e^{\psi(x,s,v)-\psi(x,\tau,v)}\times$$
$$\times\left\{\frac{1}{2}\int_0^x K(x-y,y)|v(x-y,s)|\cdot|v(y,s)-u(y,s)|\,dy\right\}dx\,ds$$
$$\le \int_0^\tau e^{LR(\tau-s)}\int_0^\infty\int_0^\infty \frac{\varphi(x)+\varphi(y)}{2}K(x,y)|v(x,s)|$$
$$\cdot|v(y,s)-u(y,s)|\,dy\,dx\,ds$$
$$\le M\int_0^\tau e^{LR(\tau-s)}\int_0^\infty\int_0^\infty \varphi(x)|v(x,s)|\cdot|v(y,s)-u(y,s)|\,dy\,dx\,ds$$
$$\le M\int_0^\tau\Big(\int_0^\infty \varphi(x)|v(x,s)|\,dx\Big)\Big(\int_0^\infty \varphi(y)|v(y,s)-u(y,s)|\frac{1}{\varphi(y)}\,dy\Big)\cdot$$
$$\cdot e^{LR(\tau-s)}\,ds$$
$$\le M\|v\|_1\|v-u\|_1 B\tau e^{L\tau R} \le LR\tau e^{LR\tau}\|u-v\|_1 \tag{4.36}$$

Using the transformation $x-y=x';\ y=y'$ and the inequality (4.22), the last term

in the equality (4.32) can be written as:

$$\begin{aligned}\int_0^\infty \varphi(x)\cdot(5)\,dx &\le \frac{1}{2}\int_0^\tau e^{\psi(x,s,v)-\psi(x,\tau,v)}\int_0^\infty\int_0^x \varphi(x)K(x-y,y)|u(y,s)|\times\\ &\times|v(x-y,s)-u(x-y,s)|\,dy\,dx\,ds\\ &\le \frac{M}{2}\int_0^\tau e^{LR(\tau-s)}\int_0^\infty\int_0^\infty[\varphi(x)+\varphi(y)]|u(y,s)|\cdot\\ &\cdot|v(x,s)-u(x,s)|\,dy\,dx\,ds\\ &\le MB\|u\|_1\|v-u\|_1\int_0^\tau e^{LRs}\,ds \le MBR\|u-v\|_1\tau e^{LR\tau}\\ &= LR\tau e^{LR\tau}\|u-v\|_1 \qquad (4.37)\end{aligned}$$

Substituting the above estimations (4.33), (4.34), (4.35), (4.36), (4.37) into the equality (4.32) we obtain

$$\begin{aligned}\|G(u)(\tau)-G(v)(\tau)\|_X &\le \int_0^\infty \varphi(x)\cdot[(1)+(2)+(3)+(4)+(5)]\,dx\\ &\le (L\tau e^{LR\tau}\|c_0\|_X + L\tau^2 e^{LR\tau}C_1(T)+(LR\tau)^2e^{LR\tau}+2LR\tau e^{LR\tau})\\ &\times\|u-v\|_1\\ &= e^{LR\tau}L\tau\Big\{\|c_0\|_X+\tau C_1(T)+LR^2\tau+2R\Big\}\|u-v\|_1\\ &\le \delta\|u-v\|_1, \qquad (4.38)\end{aligned}$$

where $\delta = e^{LR\tau}\tau L\big(\|c_0\|_X+\tau C_1(T)+LR^2\tau+2R\big)$. By (4.20), we have $\delta<1$ and therefore from (4.38) if we take $\sup_{0\le\tau\le\tau_0}$ we obtain:

$$\|G(u)-G(v)\|_1 \le \delta\|u-v\|_1$$

Therefore, relation (4.28) is proved. Thus, G is a contraction mapping in the closed ball Y_{R,τ_0} and the proof of Lemma 4.4 is complete. □

We now begin the proof of Theorem 4.1, by first proving it for a sufficiently small $T>0$. Using Lemma 4.3 and Lemma 4.4 and the Contraction Mapping Theorem (Theorem 2.1, Chapter 2), we may conclude that there exists a unique solution c to the equation (4.17) in Y_{R,τ_0} (this is what is meant by saying that there exists a unique solution locally, i.e., on the time interval $[0,\tau_0]$).

Next, we prove that the above local solution has the property that $c(x,\tau)\ge 0$, for any $x\in[0,\infty), \tau\in[0,\tau_0]$.

Lemma 4.5. The nonlinear operator G satisfies:

$$G(Y_{\tau_0}^+)\subseteq Y_{\tau_0}^+$$

Proof. Let $f \in Y_{\tau_0}^+$ be arbitrary. Then

$$G(f)(x,\tau) = c_0(x)e^{-\psi(x,\tau,f)} + \int_0^\tau e^{\psi(x,s,f)-\psi(x,\tau,f)} g(x,s)\,ds + \int_0^\tau e^{\psi(x,s,f)-\psi(x,\tau,f)} N(x,s,f)\,ds \geq 0$$

since

$$N(x,s,f) = \frac{1}{2}\int_0^x K(x-y,y)f(x-y,s)f(y,s)\,dy \geq 0$$

Therefore, $G(Y_{R,\tau_0} \cap Y_{\tau_0}^+) \subseteq Y_{R,\tau_0} \cap Y_{\tau_0}^+$ and G is a contraction mapping on $Y_{R,\tau_0} \cap Y_{\tau_0}^+$. Hence, according to the Contraction Mapping Theorem (Theorem 2.1), G has a unique fixed point c in $Y_{R,\tau_0} \cap Y_{\tau_0}^+$, which means $G(c)(x,\tau) = c(x,\tau)$ and therefore $c(x,\tau) \geq 0$. The proof of Lemma 4.5 is now complete. □

Using Lemma 4.3, Lemma 4.4 and Lemma 4.5 we may conclude that the equation (4.17) has a unique, non-negative solution $c \in C([0,\tau_0]; L^1(0,\infty))$. Next, we prove that (4.10) holds for any $\tau \in [0,\tau_0]$.

Indeed, since c satisfies (4.17) then

$$c(x,\tau) = c_0(x) + \int_0^\tau g(x,s)\,ds + \frac{1}{2}\int_0^\tau \int_0^x K(x-y,y)c(x-y,s)c(y,s)\,dy\,ds - \int_0^\tau c(x,s)\int_0^\infty K(x,y)c(y,s)\,dy\,ds \tag{4.39}$$

Let us multiply (4.39) by $\varphi(x)$. Integrating $\varphi(x)c(x,\tau)$ with respect to x over $[0,\infty)$ and using in the third integral the substitutions: $x-y=x'$ and $y=y'$, the last two integrals become

$$\begin{aligned}
&\frac{1}{2}\int_0^x \int_0^\infty \varphi(x)K(x-y,y)c(x-y,\tau)c(y,\tau)\,dy\,dx \\
&-\int_0^\infty \int_0^\infty \varphi(x)K(x,y)c(x,\tau)c(y,\tau)\,dy\,dx \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \big[\varphi(x+y)-\varphi(x)-\varphi(y)\big]K(x,y)c(x,\tau)c(y,\tau)\,dy\,dx \leq 0, \quad \text{(using (2.3))}
\end{aligned}$$

Therefore, we have

$$\int_0^\infty \varphi(x)c(x,\tau)\,dx \leq \int_0^\infty \varphi(x)c_0(x)\,dx + \int_0^\tau \int_0^\infty \varphi(x)g(x,s)\,dx\,ds, \quad \text{for any } \tau \in [0,\tau_0]$$

Hence, relation (4.10) holds for any $\tau \in [0,\tau_0]$. Moreover, in terms of X-norm, this is equivalent to

$$\|c(\tau)\|_X \leq \|c_0\|_X + TC_1(T) = \frac{R}{2} < R < \infty.$$

as $\tau_0 < T$. Taking $\sup_{0\le\tau\le\tau_0}$ we obtain

$$\sup_{0\le\tau\le\tau_0} \|c(\tau)\|_X < \infty$$

which means that $c \in L^\infty(0,\tau;X)$ Therefore, we may conclude that $c(x,\tau)$ is indeed the unique solution of the equation (4.1) in the sense of Definition 4.1 , for all $\tau \in [0,\tau_0]$ and for a.e. $x \ge 0$. □

Next, we prove the global existence of a non-negative solution to the initial value problem (4.1), (4.2) on the whole interval $[0,\infty)$ by means of a contradiction argument. Indeed, let us denote by:

$$T_0 := \sup\Big\{\tau \ge 0 \text{ s.t. the I.V.P. (4.1), (4.2) has a non-negative solution on } [0,\tau]\Big\}.$$

We want to show that $T_0 = \infty$. Since we have proved that there is a unique solution on some small $[0,\tau_0]$, we have $T_0 > 0$. Let us assume for a contradiction, that $T_0 < \infty$, and let c be the solution on $[0,T_0)$. Since c is bounded in the X-norm, $\|c\|_X < \infty$, it follows that the following limit

$$\lim_{\tau\to T_0^-} c(x,\tau)$$

exists. Therefore, by the local existence result already proved, there exists $\varepsilon > 0$, such that there exists a solution on $[T_0, T_0+\varepsilon)$, to the initial value problem (4.1), with the initial condition

$$c_0(x) = \lim_{\tau\to T_0^-} c(x,\tau). \tag{4.40}$$

In other words, we can extend the solution of the initial value problem (4.1), with the initial condition (4.40) to the right of T_0, contradicting the definition of T_0. Therefore, $T_0 = \infty$, as required.

We now prove that the solution c of the initial value problem (4.1), (4.2) is unique in the Banach space X.

Indeed, let us assume that c and d are two solutions of the equation (4.1) having the same initial value: $c_0(x) = d_0(x)$. Consider now their difference in absolute value, multiply it by $\varphi(x)$ and denote by

$$W(\tau) := \int_0^\infty \varphi(x)|c(x,\tau) - d(x,\tau)|\,dx \text{ or}$$
$$W(\tau) := \|c(\tau) - d(\tau)\|_X$$

We have

$$
\begin{aligned}
&W(\tau) \leq \\
&\leq \int_0^\tau \Big\{ \int_0^x \frac{\varphi(x)}{2} \int_0^\infty K(x-y,y)|c(x-y,s)-d(x-y,s)|\big(c(y,s)+d(y,s)\big)\,dy\,dx \\
&+ \int_0^\infty \varphi(x)|c(x,s)-d(x,s)| \int_0^\infty K(x,y)c(y,s)\,dy\,dx \\
&+ \int_0^\infty \varphi(x)d(x,s) \int_0^\infty K(x,y)|c(y,s)-d(y,s)|\,dy\,dx \Big\}\,ds \\
&= \int_0^\tau \Big\{ \int_0^\infty \int_0^\infty \frac{\varphi(x+y)}{2} K(x,y)|c(x,s)-d(x,s)|\big(c(y,s)+d(y,s)\big)\,dy\,dx \\
&+ \int_0^\infty \int_0^\infty \varphi(x)|c(x,s)-d(x,s)|K(x,y)c(y,s)\,dy\,dx \\
&+ \int_0^\infty \int_0^\infty \varphi(x)d(x,s)K(x,y)|c(y,s)-d(y,s)|\,dy\,dx \Big\}\,ds
\end{aligned}
$$

Using the subadditivity property (2.3) of φ and the hypothesis (4.5) on the coagulation kernel K we obtain:

$$
\begin{aligned}
&W(\tau) \leq \\
&\leq M \int_0^\tau \Big(\int_0^\infty \varphi(x)|c(x,s)-d(x,s)|\,dx \Big) \Big(\int_0^\infty \varphi(y)\big(c(y,s)+d(y,s)\big)\,dy \frac{1}{\varphi(y)} \Big)\,ds \\
&+ M \int_0^\tau \Big(\int_0^\infty \varphi(x)|c(x,s)-d(x,s)|\,dx \Big) \Big(\int_0^\infty \varphi(y)c(y,s) \frac{1}{\varphi(y)}\,dy \Big)\,ds \\
&+ M \int_0^\tau \Big(\int_0^\infty \varphi(x)d(x,s)\,dx \Big) \Big(\int_0^\infty \varphi(y)|c(y,s)-d(y,s)|\,dy \frac{1}{\varphi(y)} \Big)\,ds
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
W(\tau) &\leq MB \int_0^\tau \|c(s)-d(s)\|_X \Big(\|c(s)\|_X + \|d(s)\|_X \Big)\,ds \\
&+ MB \int_0^\tau \|c(s)-d(s)\|_X \|c(s)\|_X\,ds \\
&+ MB \int_0^\tau \|c(s)-d(s)\|_X \|d(s)\|_X\,ds \\
&= 2L \int_0^\tau W(s) \Big(\|c(s)\|_X + \|d(s)\|_X \Big)\,ds
\end{aligned}
$$

Since $c, d \in X$ are both solutions of (4.1), (4.2) in the sense of Definition 4.1 it will

result that:

$$\|c(s)\|_X \leq \sup_{0\leq s\leq T} \|c(s)\|_X < R_1 < \infty$$
$$\|d(s)\|_X \leq \sup_{0\leq s\leq T} \|d(s)\|_X < R_2 < \infty$$

Thus, there exists $R_1, R_2 \in (0,\infty)$ such that $\|c(s)\|_X < R_1$ and $\|d(s)\|_X < R_2$. Therefore,

$$W(\tau) \leq 2LR \int_0^\tau W(s)\, ds, \text{ where } R = R_1 + R_2 \in (0,\infty) \text{ and } \forall \tau \in [0,T]$$

Applying Grönwall's Lemma (Lemma 2.1, Chapter 2) we get: $W(\tau) \leq 0$ and, since $W(\tau) \geq 0$ we obtain therefore: $W(\tau) = 0$, with initial value $W(0) = 0$. Therefore, it will result that $W(\tau) = 0$, $\forall \tau \in [0,T]$.
From the continuity of $c(x,\tau)$ and $d(x,\tau)$ on the interval $[0,T]$ it then follows that

$$c(x,\tau) = d(x,\tau)\ \forall \tau \in [0,T],\ \forall x \in [0,\infty)$$

Therefore, $c \equiv d$ and by the arbitrariness of T the uniqueness holds for all $x \in [0,\infty)$ and $\tau \in [0,\infty)$ in X and the proof of Theorem 4.1 is now complete. □

Conclusion
In this chapter we have proved the global existence and uniqueness of solutions to the coagulation equation with source terms (4.1) with initial condition (4.2) for all time $\tau \geq 0$, in the Banach space X and for kernels that are bounded independently of their sizes. For local existence and uniqueness we have used a contraction mapping argument. The global existence and uniqueness were proved by using a contradiction argument and a well-known result in ordinary differential equation (Grönwall's inequality).

Theorem 4.1 will also be used in the next chapter, when we consider coagulation kernels which exhibit some growth.

Chapter 5

Unbounded kernels: Existence of solutions

Let us consider the following coagulation equation with source term:

$$\frac{\partial c}{\partial \tau}(x,\tau) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,\tau)c(y,\tau)\,dy - \int_0^\infty K(x,y)c(x,\tau)c(y,\tau)\,dy + g(x,\tau), \tag{5.1}$$

subject to initial condition

$$c(x,0) = c_0(x) \tag{5.2}$$

The main objective in this chapter is to prove the global existence of solutions to the coagulation equation (5.1) with a source term, with initial condition (5.2), under certain hypotheses on the growth of the coagulation kernel K in suitably defined Banach spaces. Our requirements on the initial data c_0 are less restrictive compared to those conditions imposed, for example, by Galkin in [21], who requires very strong decay conditions on the initial data. Moreover, the solutions we obtain in this chapter are in a more general Banach space and are less complicated. For physical reasons, the coagulation kernel is assumed to satisfy the same hypotheses (4.3) and (4.4) mentioned in Chapter 4.

Let

$$f(x) = \alpha + \beta x \text{ for } \alpha, \beta \geq 0 \text{ non-negative constants.}$$

In this chapter we are concerned with the coagulation kernels of the form:

$$K(x,y) = f(x)f(y), \text{ for all } (x,y) \in \mathbb{R}^2_+$$

where $f(x)$ is defined above. This kernel includes both the constant kernel and product kernel as particular cases.

We assume now that $\alpha > 0$, then the equation (5.1) becomes:

$$\frac{\partial c}{\partial \tau}(x,\tau) = \frac{1}{2}\int_0^x \alpha^2\left[1+\frac{\beta}{\alpha}(x-y)\right]\left(1+\frac{\beta}{\alpha}y\right)c(x-y,\tau)c(y,\tau)\,dy$$
$$-\int_0^\infty \alpha^2\left(1+\frac{\beta}{\alpha}x\right)\left(1+\frac{\beta}{\alpha}y\right)c(x,\tau)c(y,\tau)\,dy + g(x,\tau)$$

If we let

$$c(x,\tau) := \frac{u(x,\tau)}{\alpha}$$
$$\theta(x) := 1+\frac{\beta}{\alpha}x,\ x \geq 0$$
$$\phi(x,y) := \theta(x)\theta(y), \forall x, y \geq 0 \tag{5.3}$$

then the equation above takes the following form

$$\frac{\partial u}{\partial \tau}(x,\tau) = \frac{\alpha}{2}\int_0^x \phi(x-y,y)u(x-y,\tau)u(y,\tau)\,dy$$
$$-\alpha\int_0^\infty \phi(x,y)u(x,\tau)u(y,\tau)\,dy + \alpha g(x,\tau) \text{ a.e. } (x \geq 0, \tau \geq 0) \tag{5.4}$$

subject to the initial condition

$$u(x,0) = u_0(x) = \alpha c_0(x) \text{ a.e. } (x \geq 0) \tag{5.5}$$

Therefore, the new coagulation kernel ϕ has the same properties as the old coagulation kernel K.

In order to study the existence of solutions to the equation (5.4) with initial condition (5.5), we introduce the following Banach space:

$$X := L^1(0,\infty;\theta(x)\,dx) \tag{5.6}$$

endowed with the norm $\|\cdot\|_X$ defined by

$$\|f\|_X = \int_0^\infty \left(1+\frac{\beta}{\alpha}x\right)|f(x)|\,dx \qquad \text{for } f \in X \tag{5.7}$$

In other words,

$$X = \left\{f \in L^1(0,\infty) : \|f\|_X < \infty\right\}$$

Let X^+ be the positive cone of the Banach space X, i.e. $X^+ = \{f \in X : f \geq 0 \text{ a.e.}\}$. We also assume that the initial data $u_0 \in X^+$, and we set

$$C_0 := \|u_0\|_X \in [0,\infty) \tag{5.8}$$

Let $T \in (0,\infty]$ be arbitrary.

Our first step is to investigate the problem of existence of solutions to the coagulation

equation (5.4) for the class of kernels $\phi(x,y)$ defined in (5.3), with the initial data $u_0 \in X^+$ satisfying (5.8), and where the source term $g(x,\tau)$ satisfies the following hypotheses:
(A)

$$\int_0^\infty \theta(x)g(x,\tau)\,dx \in L^1(0,T) \text{ for almost all } \tau. \tag{5.9}$$

(B) There exists a strictly positive constant $C_1(T) \in (0,\infty)$ such that

$$\text{ess sup}_{0\le\tau\le T} \int_0^\infty \theta(x)g(x,\tau)\,dx = C_1(T) \tag{5.10}$$

Before stating any result let us define the notion of solution of the coagulation equation (5.4), (5.5) that will be used further:

Definition 5.1. Let $T \in (0,\infty]$ be arbitrary. A solution u of the equation (5.4) is a function $u : [0,T) \to X^+$ such that, for every $\tau \in [0,T)$, there holds

(a) $u \in C([0,T]; L^1(0,\infty)) \cap L^\infty(0,\tau;X)$

(b)

$$(x,y,s) \mapsto \phi(x,y)u(x,s)u(y,s) \in L^1\big((0,\infty)\times(0,\infty)\times(0,T)\big) \tag{5.11}$$

(c)

$$\int_0^\infty \phi(x,y)u(y,\tau)\,dy \in L^1(0,T) \text{ a.e. } (x \ge 0, \tau \ge 0)$$

(d) For almost every $x \in [0,\infty)$:

$$u(x,\tau) = u_0(x) + \int_0^\tau \alpha g(x,s)ds + \frac{1}{2}\int_0^\tau \int_0^x \alpha\phi(x-y,y)u(x-y,s)u(y,s)\,dy\,ds$$
$$- \int_0^\tau u(x,s)\int_0^\infty \alpha\phi(x,y)u(y,s)\,dy\,ds \tag{5.12}$$

with u_0 and g defined in (5.8), and (5.9), (5.10), respectively.

Remark: It follows from (5.12) above that for almost all $x \in [0,\infty)$, $u(x,\cdot)$ is absolutely continuous on $[0,T)$ so that for a.e. $x \ge 0$, $u(x,\tau)$ satisfies (5.4) for a.e. $\tau \in [0,T)$.

Our main result is the following theorem which provides the global existence of a solution to the equation (5.4) (in the sense of Definition 5.1) for any initial data $u_0 \in X^+$, without using any explicit representation for u_0.

Theorem 5.1. (Global existence of solutions of (5.4), (5.5)) Assume that the coagulation kernel ϕ is as in (5.3) and the source term g satisfies (5.9) and (5.10).

For every $u_0 \in X^+$ satisfying (5.8), there exists at least one solution $u \in X^+$ to the equation (5.4) on $[0,T)$ for every $T \in (0,\infty]$ with $u(0) = u_0$ satisfying

$$\int_0^\infty xu(x,\tau)\,dx \le \int_0^\infty xu_0(x)\,dx + \alpha \int_0^\tau \int_0^\infty xg(x,s)\,dx\,ds \qquad \text{for all } \tau \in [0,T] \tag{5.13}$$

In order to prove Theorem 5.1, we have to study first the properties of the solutions of the approximating equations. The main goal is to show the weak compactness in $L^1(0,\infty)$ of the sequence of solutions to the approximating equations.

Approximating equations

We prove the existence of solutions of (5.4), (5.5) by taking the limit of a sequence of approximating equations obtained by replacing the kernel ϕ in (5.3) by the so-called "cut-off" kernels ϕ_n, $n \in \mathbb{N}^*$. Let $\{\xi_n\}_{n\ge1}$ be the sequence of cut-off functions defined by

$$\xi_n(x) := \begin{cases} 1, & \text{if } 0 \le x \le n \\ 0, & \text{if } x \ge n+1 \text{ or } x < 0 \end{cases}$$

For $n \ge 1$, we define a sequence ϕ_n of approximations of ϕ by:

$$\phi_n(x,y) := \phi(x,y)\xi_n(x)\xi_n(y),\ \forall (x,y) \in \mathbb{R}^2_+ \tag{5.14}$$

Thus

$$\phi_n(x,y) = \begin{cases} \phi(x,y), & \text{if } x,y \in [0,n] \\ 0, & \text{if } x,y \ge n+1 \text{ or (if either of } x \text{ or } y < 0) \end{cases}$$

Using the properties of ξ_n, and as a consequence of (5.3), we obtain the following:

Lemma 5.1. For each fixed $n \in \mathbb{N}, n \ge 1$, the function ϕ_n is a non-negative, continuous and bounded function on $\mathbb{R}^2_+$ and satisfies:

$$\phi_n(x,y) = \theta_n(x)\theta_n(y), \text{ for } (x,y) \in \mathbb{R}^2_+, \tag{5.15}$$

where $\theta_n(x) = \theta(x)\xi_n(x)$, $\forall x \in \mathbb{R}_+$ and $n \ge 1$. □

Consider the following approximating equations, which result from (5.4) by replacing ϕ with the cut-off kernels ϕ_n:

$$\begin{aligned} \frac{\partial u_n}{\partial \tau}(x,\tau) &= \frac{\alpha}{2}\int_0^x \phi_n(x-y,y)u_n(x-y,\tau)u_n(y,\tau)\,dy \\ &\quad - \int_0^\infty \alpha\phi_n(x,y)u_n(x,\tau)u_n(y,\tau)\,dy + \alpha g(x,\tau) \qquad \text{a.e. } x,\tau \ge 0 \\ u_n(x,0) &= u_0(x) \text{ the initial condition, } \forall x \ge 0 \end{aligned} \tag{5.16}$$

According to Lemma 5.1, the kernel ϕ_n is bounded and continuous on $\mathbb{R}^2_+$. Therefore, we may apply Theorem 4.1, Chapter 4, in order to establish the existence and uniqueness of a solution to the equation (5.16) with ϕ replaced by ϕ_n. Moreover, Theorem 4.1 holds for any weight function φ satisfying properties (2.2), (2.3). In particular, it also holds for $\varphi \equiv \theta$. More precisely, the following corollary of Theorem 4.1 holds

Corollary 5.1. For each $n \geq 1$, $u_0 \in X^+$, and g satisfying (5.9), (5.10), there exists a unique function $u_n \in C([0,T]; L^1(0,\infty)) \cap L^\infty(0,\tau; X)$ with $T \in (0,\infty]$ such that, for every $(x,\tau) \in \mathbb{R}^2_+$ there holds:

$$u_n(x,\tau) = u_0(x) + \frac{1}{2}\int_0^\tau \int_0^x \alpha\phi_n(x-y,y)u_n(x-y,s)u_n(y,s)\,dy\,ds - \int_0^\tau \alpha u_n(x,s)\int_0^\infty \phi_n(x,y)u_n(y,s)\,dy\,ds + \int_0^\tau \alpha g(x,s)\,ds \tag{5.17}$$

Since the coagulation kernel ϕ_n is compactly supported in $\mathbb{R}^2_+$, from (5.17) we obtain the following identities which prove to be very helpful later in this chapter.

Lemma 5.2. Let L be a locally bounded function on $(0,\infty)$ such that $L(x) \leq G(1+x)$, for some constant $G > 0$. Let $\alpha > 0$ be a positive constant. For each $n \geq 1$, $\tau \in (0,\infty)$, and $s \in [0,\tau)$ there holds:

$$\int_0^\infty L(x)\big[u_n(x,\tau) - u_n(x,s)\big]\,dx = \frac{1}{2}\int_s^\tau \int_0^\infty \int_0^\infty \alpha\phi_n(x,y)\widetilde{L}(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma + \int_s^\tau \int_0^\infty \alpha L(x)g(x,\sigma)\,dx\,d\sigma \tag{5.18}$$

$$\int_0^\infty x u_n(x,\tau)\,dx \leq \int_0^\infty x u_0(x)\,dx + \int_0^\tau \int_0^\infty \alpha x g(x,\sigma)\,dx\,d\sigma \tag{5.19}$$

where

$$\widetilde{L}(x,y) = L(x+y) - L(x) - L(y) \qquad \text{for } (x,y) \in \mathbb{R}^2_+. \tag{5.20}$$

Proof Since u_n satisfies the integral equation (5.17), then multiplying equation (5.17) by $L(x)$ and integrating over $x \in [0,\infty)$ we have

$$\int_0^\infty L(x)u_n(x,\tau)\,dx = \frac{1}{2}\int_0^\tau \int_0^\infty \int_0^\infty \alpha\phi_n(x,y)L(x+y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma - \int_0^\tau \int_0^\infty \int_0^\infty \alpha\phi_n(x,y)u_n(x,\sigma)u_n(y,\sigma)L(x)\,dx\,dy\,d\sigma + \int_0^\tau \int_0^\infty \alpha L(x)g(x,\sigma)\,dx\,d\sigma + \int_0^\infty u_0(x)L(x)\,dx$$

where, in the first integral, we have used Fubini's theorem, the symmetry property of ϕ_n and the substitutions $x - y = x'; y = y'$. Next, subtract the corresponding equation in $u_n(x,s)$ from the above to get:

$$\int_0^\infty L(x)\big[u_n(x,\tau) - u_n(x,s)\big]\,dx = \frac{1}{2}\int_s^\tau \int_0^\infty \int_0^\infty \alpha\phi_n(x,y)L(x+y)$$
$$\times\, u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma - \int_s^\tau \int_0^\infty \int_0^\infty \alpha\phi_n(x,y)u_n(x,\sigma)u_n(y,\sigma)L(x)\,dx\,dy\,d\sigma$$
$$+ \int_s^\tau \int_0^\infty \alpha L(x)g(x,\sigma)\,dx\,d\sigma$$
$$= \frac{1}{2}\int_s^\tau \int_0^\infty \int_0^\infty \alpha\phi_n(x,y)\big[L(x+y) - L(x) - L(y)\big]u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma$$
$$+ \int_s^\tau \int_0^\infty \alpha L(x)g(x,\sigma)\,dx\,d\sigma$$

therefore the equality (5.18) holds. When we replace in (5.18) $L(x)$ by x and $s = 0$, relation (5.19) holds and therefore the proof of Lemma 5.2 is complete. □

In the following we use the special form (5.14) of the coagulation kernel ϕ_n to derive some estimates which hold uniformly with respect to $n \geq 1$. We denote by $C_i(a,T)_{i\geq 2}$ or $C_i(T)_{i\geq 2}$ subsequent positive constants which depend only on $C_0, g, \phi, \theta, \alpha > 0, \beta > 0, a$. The dependence of C_i upon further parameters will be indicated explicitely.

Lemma 5.3. For $P > 0$ and $n \geq P$ there holds

$$\int_0^\tau \int_P^\infty \int_P^\infty \phi_n(x,y)u_n(x,s)u_n(y,s)\,dx\,dy\,ds \leq \frac{2}{\beta P}\big(C_0 + \alpha T C_1(T)\big) \tag{5.21}$$

Proof. Let $L(x) = min\{x, P\}$ for all x. From (5.18), it follows that if $x \in [0,P]$ or $y \in [0,P]$, then $\widetilde{L}(x,y) = L(x+y) - L(x) - L(y) \leq 0$, while if $(x,y) \in [P,\infty)\times[P,\infty)$, then $\widetilde{L}(x,y) = -P$. From (5.18) when $s \equiv 0$:

$$\int_0^\infty L(x)\big[u_n(x,\tau) - u_0(x)\big]\,dx$$
$$= \frac{1}{2}\int_0^\tau \int_0^\infty \int_0^\infty \alpha\phi_n(x,y)\widetilde{L}(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma$$
$$+ \int_0^\tau \int_0^\infty \alpha L(x)g(x,\sigma)\,dx\,d\sigma$$

Let us denote by $\mathcal{R}_P = [P,\infty)\times[P,\infty)$. Since for x or $y \in [0,P]$ we have $\widetilde{L}(x,y) \leq 0$, then:

$$\int\int_{\mathbb{R}^2\setminus\mathcal{R}_P} \phi_n(x,y)\widetilde{L}(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy \leq 0$$

and so

$$\int_0^\infty \int_0^\infty \phi_n(x,y)\widetilde{L}(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy \leq \int\int_{\mathcal{R}_P} \phi_n(x,y)\widetilde{L}(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy. \tag{5.22}$$

Thus

$$\begin{aligned}
&\int_0^\infty L(x)\big[u_n(x,\tau)-u_0(x)\big]\,dx \\
&\quad\leq \frac{1}{2}\int_0^\tau \int_P^\infty \int_P^\infty \alpha\phi_n(x,y)\widetilde{L}(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\
&\quad+\int_0^\tau \int_0^\infty \alpha L(x)g(x,\sigma)\,dx\,d\sigma \\
&\quad\leq -\frac{P}{2}\int_0^\tau \int_P^\infty \int_P^\infty \alpha\phi_n(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\
&\quad+\alpha\int_0^\tau \int_0^\infty g(x,\sigma)L(x)\,dx\,d\sigma
\end{aligned}$$

Passing the first integral in the right-hand side to the left-hand side we get:

$$\begin{aligned}
&\frac{P}{2}\int_0^\tau \int_P^\infty \int_P^\infty \alpha\phi_n(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\
&\quad\leq \int_0^\infty xu_0(x)\,dx+\alpha\int_0^\tau \int_0^\infty xg(x,\sigma)\,dx\,d\sigma \\
&\quad=\frac{\alpha}{\beta}\int_0^\infty (1+\frac{\beta}{\alpha}x)u_0(x)\,dx+\frac{\alpha^2}{\beta}\int_0^\tau \int_0^\infty (1+\frac{\beta}{\alpha}x)g(x,\sigma)\,dx\,d\sigma \\
&\quad-\frac{\alpha}{\beta}\int_0^\infty u_0(x)\,dx-\frac{\alpha^2}{\beta}\int_0^\tau \int_0^\infty g(x,\sigma)\,dx\,d\sigma \\
&\quad\leq \frac{\alpha}{\beta}\|u_0\|_X+\frac{\alpha^2}{\beta}TC_1(T)=\frac{\alpha}{\beta}C_0+\frac{\alpha^2}{\beta}TC_1(T)
\end{aligned}$$

Dividing this relation by $\frac{\alpha P}{2}$ we get (5.21). □

Lemma 5.4. Let $T\in(0,\infty)$. For $\tau\in[0,T]$ and $n\geq P\geq 1$ there holds:

$$\int_0^\tau \Big(\int_P^\infty \theta_n(x)u_n(x,s)\,dx\Big)^2\,ds \leq \frac{2}{\beta P}(C_0+\alpha TC_1(T)) \tag{5.23}$$

Proof

$$\begin{aligned}
&\int_0^\tau \Big(\int_P^\infty \theta_n(x) u_n(x,s)\, dx \Big)^2 ds \\
&\quad = \int_0^\tau \Big(\int_P^\infty \theta_n(x) u_n(x,s)\, dx \Big) \Big(\int_P^\infty \theta_n(y) u_n(y,s)\, dy \Big)\, ds \\
&\quad = \int_0^\tau \int_P^\infty \int_P^\infty \theta_n(x)\theta_n(y) u_n(x,s) u_n(y,s)\, dx\, dy\, ds \\
&\quad = \int_0^\tau \int_P^\infty \int_P^\infty \phi_n(x,y) u_n(x,s) u_n(y,s)\, dx\, dy\, ds \\
&\quad \le \frac{2}{\beta P}(C_0 + \alpha T C_1(T)) \qquad \text{(using (5.21))}
\end{aligned}$$

where $\theta_n(x) = \theta(x)\xi_n(x)$, $\forall x \in \mathbb{R}_+$, for $n \ge 1$. Therefore, Lemma 5.4 is proved.□

Before going further we introduce the following notation for $n \ge 1$, $a \in (1, \infty]$, $\delta \in (0, \infty)$ and $\tau \in \mathbb{R}_+$. Let $\varepsilon > 0$ and $A \subset \mathbb{R}_+$ be a measurable subset of $\mathbb{R}_+$ with $\mu(A) \le \delta$.
Let us denote by:

$$q^n_{a,\delta}(\tau) := \sup_{A \subset \mathbb{R}_+} \int_0^a \chi_A(x) u_n(x,\tau)\, dx\ , \tag{5.24}$$

where the supremum is taken over all measurable subsets A of $\mathbb{R}_+$ such that $\mu(A) \le \delta$ (where $\mu(A)$ denotes the Lebesgue measure of A). Here, $\chi_A(x)$ is the characteristic function of A:

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

Lemma 5.5. Let $T \in (0, \infty)$ and $a \in (1, \infty)$. For every $n \ge 1, \tau \in [0, T]$ and $\delta \in (0, \infty)$ there holds:

$$\int_0^\infty u_n(x,\tau)\, dx \le C_2(T) \tag{5.25}$$

$$\int_0^\tau \Big(\int_0^\infty \theta_n(x) u_n(x,s)\, dx \Big)^2 ds \le \frac{2}{\alpha} C_2(T) \tag{5.26}$$

$$q^n_{a,\delta}(\tau) \le C_6(a,T)\big(q^n_{a,\delta}(0) + \delta \alpha T C_1(T)\big) \tag{5.27}$$

where $C_2(T)$, $C_6(a,T)$ are strictly positive constants (defined below).

Proof Let $L \equiv 1$ in (5.18), so $\widetilde{L} \equiv -1$. From (5.18) with $s \equiv 0$ it will result

$$\begin{aligned}
&\int_0^\infty [u_n(x,\tau) - u_0(x)]\, dx = -\frac{1}{2} \int_0^\tau \int_0^\infty \int_0^\infty \alpha \phi_n(x,y) u_n(x,\sigma) u_n(y,\sigma)\, dx\, dy\, d\sigma \\
&+ \int_0^\tau \int_0^\infty \alpha g(x,\sigma)\, dx\, d\sigma
\end{aligned}$$

Passing $-\frac{1}{2}\int_0^\tau\int_0^\infty\int_0^\infty \alpha\phi_n(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma$ to the left-hand side and $\int_0^\infty u_0(x)\,dx$ to the right, we get

$$\int_0^\infty u_n(x,\tau)\,dx + \frac{1}{2}\int_0^\tau\int_0^\infty\int_0^\infty \alpha\phi_n(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma = \int_0^\infty u_0(x)\,dx + \int_0^\tau\int_0^\infty \alpha g(x,\sigma)\,dx\,d\sigma \text{ (denote this equality by(*))}$$

so, using (5.10) we have

$$\begin{aligned}\int_0^\infty u_n(x,\tau)\,dx &\le -\frac{1}{2}\int_0^\tau\int_0^\infty\int_0^\infty \alpha\phi_n(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\ &\quad + \int_0^\infty \theta(x)u_0(x)\,dx + \alpha\int_0^\tau\Big(\int_0^\infty \theta(x)g(x,\sigma)\,dx\Big)\,d\sigma \\ &\le \|u_0\|_X + \alpha\int_0^\tau \operatorname{ess\,sup}_{0\le\sigma\le T}\|g(\sigma)\|_X\,d\sigma \\ &\le C_0 + \alpha T C_1(T) =: C_2(T) \qquad \text{(notation)}\end{aligned}$$

therefore (5.25) holds true. Also from the equality (*) we have

$$\frac{\alpha}{2}\int_0^\tau\int_0^\infty\int_0^\infty \theta_n(x)\theta_n(y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \le C_2(T)$$

Divide the above inequality by $\frac{\alpha}{2}$ so,

$$\int_0^\tau\Big(\int_0^\infty \theta_n(x)u_n(x,\sigma)\,dx\Big)^2\,d\sigma \le \frac{2}{\alpha}C_2(T)$$

therefore (5.26) holds.

To prove (5.27), let $a\in(1,\infty)$, $\delta\in(0,\infty)$ and consider a measurable subset A of $\mathbb{R}_+$ with $\mu(A)\le\delta$. By the nonnegativity of $u_n\ge 0$ and $\phi_n\ge 0, \forall n\ge 1$, it follows from (5.17) that

$$\begin{aligned}u_n(x,\tau) = u_0(x) + \int_0^\tau\Big\{&\frac{1}{2}\int_0^x \alpha\phi_n(x-y,y)u_n(x-y,s)u_n(y,s)\,dy \\ &- \int_0^\infty \alpha\phi_n(x,y)u_n(x,s)u_n(y,s)\,dy + \alpha g(x,s)\Big\}\,ds \qquad (5.28)\end{aligned}$$

Multiply (5.28) by $\chi_A(x)$ and integrate over $x\in[0,a]$ to get:

$$\begin{aligned}\int_0^a \chi_A(x)u_n(x,\tau)\,dx &= \int_0^a \chi_A(x)u_0(x)\,dx + \alpha\int_0^\tau\int_0^a \chi_A(x)g(x,s)\,dx\;ds \\ &\quad + \frac{\alpha}{2}\int_0^\tau\int_0^a\int_0^x \chi_A(x)\phi_n(x-y,y)u_n(x-y,s)u_n(y,s)\,dy\,dx\,ds \\ &\quad - \alpha\int_0^\tau\int_0^a\int_0^\infty \chi_A(x)\phi_n(x,y)u_n(x,s)u_n(y,s)\,dy\,dx\,ds.\end{aligned}$$

Removing non-positive terms, we find that the above is

$$\leq \int_0^a \chi_A(x)u_0(x)\,dx + \alpha\int_0^\tau\int_0^a \chi_A(x)g(x,s)\,dx\,ds$$
$$+\frac{\alpha}{2}\int_0^\tau\int_0^a\int_0^x \chi_A(x)\phi_n(x-y,y)u_n(x-y,s)u_n(y,s)\,dy\,dx\,ds \qquad (5.29)$$

In the third integral in (5.29) use Fubini's theorem and the property of the characteristic function: $\chi_{A-y}(x) = \chi_A(x+y)$ (where we denoted by $A-y := \{r-y : r \in A\}$) and the integral becomes

$$\frac{\alpha}{2}\int_0^\tau\int_0^a\int_0^x \chi_A(x)\phi_n(x-y,y)u_n(x-y,s)u_n(y,s)\,dy\,dx\,ds$$
$$=\frac{\alpha}{2}\int_0^\tau\int_0^a\int_0^a \chi_{A-y}(x)\phi_n(x,y)u_n(x,s)u_n(y,s)\,dy\,dx\,ds$$
$$=\frac{\alpha}{2}\int_0^\tau\int_0^a\int_0^a \chi_A(x+y)\phi_n(x,y)u_n(x,s)u_n(y,s)\,dy\,dx\,ds \qquad (**)$$

Since $\phi_n(x,y)$ is uniformly bounded with respect to $n \geq 1$ on $[0,a]\times[0,a]$ we may take

$$\widetilde{\phi}_{0,n}(a) = \sup_{0\leq x,y\leq a}\frac{\alpha}{2}\phi_n(x,y) \qquad \text{and}$$
$$\widetilde{\phi}_0(a) = \max_{n\geq 1}\widetilde{\phi}_{0,n}(a)$$

Therefore, $\frac{\alpha}{2}\phi_n(x,y) \leq \widetilde{\phi}_{0,n}(a) \leq \widetilde{\phi}_0(a)$, for any $(x,y) \in [0,a]\times[0,a]$. The bound $\widetilde{\phi}_0(a)$ depends on a. Therefore (**) becomes

$$\leq \widetilde{\phi}_0(a)\int_0^\tau\int_0^a u_n(y,s)\Big(\int_0^a \chi_A(x+y)u_n(x,s)\,dx\Big)\,dy\,ds$$

(since the Lebesgue measure is invariant with respect to translation) $\Rightarrow$

$$= \widetilde{\phi}_0(a)\int_0^\tau\int_0^a u_n(y,s)\chi_A(y)\Big(\int_0^a u_n(x,s)\,dx\Big)\,dy\,ds$$
$$= \widetilde{\phi}_0(a)\int_0^\tau\Big(\int_0^a u_n(x,s)\,dx\Big)\Big(\int_0^a \chi_A(y)u_n(y,s)\,dy\Big)\,ds$$
$$\leq \widetilde{\phi}_0(a)\int_0^\tau\Big(\int_0^a u_n(x,s)\,dx\Big)\Big(\sup_{A\subset\mathbb{R}_+}\int_0^a \chi_A(y)u_n(y,s)\,dy\Big)\,ds$$
$$\leq \widetilde{\phi}_0(a)\int_0^\tau q^n_{a,\delta}(s)\Big(\int_0^\infty u_n(x,s)\,dx\Big)\,ds \leq \widetilde{\phi}_0(a)C_2(T)\int_0^\tau q^n_{a,\delta}(s)\,ds \qquad \text{(using (5.25))}$$

The first integral in (5.29) becomes

$$\int_0^a \chi_A(x)u_0(x)\,dx \leq \sup_{A\subset\mathbb{R}_+}\int_0^a \chi_A(x)u_0(x)\,dx = q^n_{a,\delta}(0)$$

Moreover, the second integral in (5.29) becomes:

$$\alpha \int_0^a \chi_A(x) g(x,s)\,dx = \begin{cases} \alpha \int_0^a g(x,s)\,dx & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$\leq \alpha\mu(A)\Big[\text{ess sup}_{0\leq s\leq T} \int_0^a g(x,s)\,dx\Big]$$

$$\leq \alpha\delta\Big[\text{ess sup}_{0\leq s\leq T} \int_0^a \theta(x) g(x,s)\,dx\Big]$$

$$\leq \delta\alpha\Big[\text{ess sup}_{0\leq s\leq T} \int_0^\infty \theta(x) g(x,s)\,dx\Big] = \delta\alpha C_1(T).$$

Therefore,

$$\alpha \int_0^\tau \int_0^a \chi_A(x) g(x,s)\,dx\,ds \leq \delta\alpha \int_0^\tau C_1(T)\,ds \leq \alpha\delta T C_1(T)$$

Therefore, (5.29) becomes

$$\int_0^a \chi_A(x) u_n(x,\tau)\,dx \leq q^n_{a,\delta}(0) + \delta\alpha T C_1(T) + C_2(T)\widetilde{\phi}_0(a) \int_0^\tau q^n_{a,\delta}(s)\,ds$$

Take now $\sup_{A\subset\mathbb{R}_+}$, and denote by

$$\alpha T C_1(T) := C_3(T) > 0,$$
$$\widetilde{\phi}_0(a) C_2(T) := C_4(a,T) > 0$$

Therefore

$$q^n_{a,\delta}(\tau) \leq q^n_{a,\delta}(0) + C_4(a,T) \int_0^\tau q^n_{a,\delta}(s)\,ds + \delta C_3(T),$$

so

$$q^n_{a,\delta}(\tau) \leq \big\{q^n_{a,\delta}(0) + \delta C_3(T)\big\} + C_4(a,T) \int_0^\tau q^n_{a,\delta}(s)\,ds$$

Denote by $C_5(a,T,\delta) := q^n_{a,\delta}(0) + \delta C_3(T) > 0$. Then

$$q^n_{a,\delta}(\tau) \leq C_5(a,T,\delta) + C_4(a,T) \int_0^\tau q^n_{a,\delta}(s)\,ds \qquad \text{(From Grönwall's Lemma 2.1)} \Rightarrow$$
$$q^n_{a,\delta}(\tau) \leq C_5(a,T,\delta) e^{C_4(a,T)\tau} \leq C_5(a,T,\delta) e^{TC_4(a,T)}$$

Denote by $e^{TC_4(a,T)} =: C_6(a,T)$ and we get

$$q^n_{a,\delta}(\tau) \leq C_6(a,T)\Big(q^n_{a,\delta}(0) + \delta\alpha T C_1(T)\Big)$$

which proves (5.27) holds and completes the proof of Lemma 5.5. □

Lemma 5.6. Let $T \in (0,\infty)$ and $a \in (1,\infty)$. For every $n \geq 1$, $\tau \in [0,T]$ and $s \in [0,\tau]$ there holds:

$$\int_0^a \big|u_n(x,\tau) - u_n(x,s)\big|\,dx \leq C_7(a,T)(\tau - s)^{\frac{1}{2}} \tag{5.30}$$

Proof. We take in (5.18)

$$L(x) = \chi_{[0,a]}(x)\, sgn\big(u_n(x,\tau) - u_n(x,s)\big),$$

so

$$L(x) = \begin{cases} \pm 1 & \text{if } x \in [0,a] \\ 0 & \text{if } x \notin [0,a] \end{cases}$$

From (5.18) with $L(x)$ given above we have

$$\begin{aligned}
&\int_0^\infty L(x)\big[u_n(x,\tau) - u_n(x,s)\big]\,dx = \int_0^a L(x)\big[u_n(x,\tau) - u_n(x,s)\big]\,dx \\
&+ \int_a^\infty \chi_{[0,a]}(x)L(x)[u_n(x,\tau) - u_n(x,s)]\,dx = \int_0^a \big|u_n(x,\tau) - u_n(x,s)\big|\,dx \\
&\leq \frac{\alpha}{2}\int_s^\tau \int_0^a \int_0^a \phi_n(x,y)|\widetilde{L}(x,y)|u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\
&+ \frac{\alpha}{2}\int_s^\tau \int_0^a \int_a^\infty \phi_n(x,y)\big(|L(x+y)| + |L(x)| + |L(y)|\big)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\
&+ \int_s^\tau \int_0^\infty \alpha|L(x)|g(x,\sigma)\,dx\,d\sigma
\end{aligned} \tag{5.31}$$

The first integral in (5.31) becomes

$$\begin{aligned}
&\frac{\alpha}{2}\int_s^\tau \int_0^a \int_0^a \phi_n(x,y)|\widetilde{L}(x,y)|u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\
&\leq \frac{\alpha}{2}\int_s^\tau \int_0^a \int_0^a \phi_n(x,y)\big[|L(x+y)| + |L(x)| + |L(y)|\big]u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\
&\leq \frac{3\alpha}{2}\int_s^\tau \int_0^a \int_0^a \phi_n(x,y)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\
&\leq 3\widetilde{\phi}_0(a)\int_s^\tau \Big(\int_0^a u_n(x,\sigma)\,dx\Big)^2 d\sigma \leq 3\widetilde{\phi}_0(a)[C_2(T)]^2(\tau - s) = C_8(a,T)(\tau - s)
\end{aligned}$$

where we have set $C_8(a,T) := 3\widetilde{\phi}_0(a)[C_2(T)]^2 > 0$.

The second integral in (5.31) may be written as

$$\begin{aligned}
&\frac{\alpha}{2}\int_s^\tau \int_0^a \int_a^\infty \phi_n(x,y)\big(|L(x+y)| + |L(x)| + |L(y)|\big)u_n(x,\sigma)u_n(y,\sigma)\,dx\,dy\,d\sigma \\
&\leq \alpha\int_s^\tau \Big(\int_a^\infty \theta_n(x)u_n(x,\sigma)\,dx\Big)\Big(\int_0^a \theta_n(y)u_n(y,\sigma)\,dy\Big)\,d\sigma \\
&\leq C_2(T)C_{10}(a)\int_s^\tau \Big(\int_0^\infty \theta_n(x)u_n(x,\sigma)\,dx\Big)\,d\sigma
\end{aligned} \tag{5.32}$$

where in (5.32) we have used Hölder's inequality and we have obtained:

$$\int_0^a \theta_n(y)u_n(y,\sigma)\,dy \le \|\theta\|_{L^\infty(0,a)} \int_0^\infty u_n(y,\sigma)\,dy < C_2(T)C_{10}(a) \text{ (using (5.25))}$$

where $C_{10}(a) := \|\theta\|_{L^\infty(0,a)} > 0$.

Next, put $C_{11}(a,T) := C_2(T)C_{10}(a) > 0$. Then, using Hölder's inequality, the last term in (5.32) takes the form:

$$\begin{aligned}
C_{11}(a,T)\int_s^\tau \Big(\int_0^\infty \theta_n(x)u_n(x,\sigma)\,dx\Big)\,d\sigma &\le C_{11}(a,T)\int_s^\tau 1\cdot\Big(\int_0^\infty \theta_n(x)u_n(x,\sigma)\,dx\Big)\,d\sigma \\
&\le C_{11}(a,T)\Big(\int_s^\tau 1^2\,d\sigma\Big)^{\frac12}\Big\{\int_s^\tau\Big(\int_0^\infty \theta_n(x)u_n(x,\sigma)\,dx\Big)^2\Big\}^{\frac12} \\
&= C_{11}(a,T)(\tau-s)^{\frac12}\Big\{\int_0^\tau\Big(\int_0^\infty \theta_n(x)u_n(x,\sigma)\,dx\Big)^2\Big\}^{\frac12} \\
&\le C_{11}(a,T)\Big(\frac{2}{\alpha}C_2(T)\Big)^{\frac12}(\tau-s)^{\frac12} \text{ (using (5.26))}
\end{aligned}$$

The third integral in (5.31) may be expressed as

$$\begin{aligned}
\int_s^\tau\int_0^\infty \alpha|L(x)|g(x,\sigma)\,dx\,d\sigma &\le \alpha\int_s^\tau \text{ess sup}_{0\le\sigma\le\tau\le T}\Big(\int_0^\infty \theta(x)g(x,\sigma)\,dx\Big)\,d\sigma \\
&= \alpha(\tau-s)C_1(T)
\end{aligned}$$

Substituting the above inequalities into (5.31) we get

$$\begin{aligned}
&\int_0^a |u_n(x,\tau)-u_n(x,s)|\,dx < C_8(a,T)(\tau-s) + C_{11}(a,T)\sqrt{\frac{2}{\alpha}C_2(T)}(\tau-s)^{\frac12} \\
&+\alpha(\tau-s)C_1(T) \le \big[C_8(a,T)+\alpha C_1(T)\big](\tau-s) + C_{11}(a,T)\sqrt{\frac{2}{\alpha}C_2(T)}(\tau-s)^{\frac12} \\
&\le C_{12}(a,T)\big[(\tau-s)+(\tau-s)^{\frac12}\big] \le C_{12}(a,T)T(\tau-s)^{\frac12} \\
&\le C_7(a,T)(\tau-s)^{\frac12}
\end{aligned}$$

where, we set $C_{12}(a,T) := \max\Big\{C_8(a,T)+\alpha C_1(T), C_{11}(a,T)\sqrt{\frac{2}{\alpha}C_2(T)}\Big\} > 0$ and $C_7(a,T) := TC_{12}(a,T) > 0$, since all constants above are strictly positive numbers. Thus, the above inequality and notation complete the proof of Lemma 5.6. □

Proposition 5.1. (Compactness result for the sequence of approximate solutions $\{u_n\}_{n\ge1}$) For each $T \in (0,\infty)$, the sequence $\{u_n\}_{n\ge1}$ is relatively sequentially compact in $C([0,T];L^1(0,\infty)_w)$ defined in (2.8).

Proof. In order to prove this proposition we apply the variant of the Arzela-Ascoli's theorem 2.4 given in the Chapter 2. More precisely, we need to prove the properties (P1) and (P2) of the theorem.

Proof of (P1) In order to prove this proposition we shall make use of the Dunford-Pettis Theorem 2.5 given in the Chapter 2 as it gives necessary and sufficient conditions for (P1) to hold. Theorem 2.5 states these conditions (see properties (P3), (P4), (P5)). Let us prove these conditions for the set $\mathcal{P} = \{u_n(\tau) : n \geq 1\}$. In other words, we prove that the set $\mathcal{P}$ is weakly relatively compact in $L^1(0,\infty)$. .

(i) Proof of (P3). We have to prove that there exists a constant $R \in (0,\infty)$ such that

$$\int_0^\infty \theta(x)u_n(x,\tau)\,dx \leq R \qquad \text{for } n \geq 1 \text{ and } \tau \in [0,T] \tag{5.33}$$

Indeed, as we have already proved in Lemma 5.2,

$$\begin{aligned}
\int_0^\infty \theta(x)u_n(x,\tau)\,dx &= \int_0^\infty \Big(1+\frac{\beta}{\alpha}x\Big)u_n(x,\tau)\,dx \\
&= \int_0^\infty u_n(x,\tau)\,dx + \frac{\beta}{\alpha}\int_0^\infty x u_n(x,\tau)\,dx \qquad \text{By (5.19) and (5.25),} \\
&\leq C_2(T) + \frac{\beta}{\alpha}\Big\{\int_0^\infty x u_0(x)\,dx + \alpha\int_0^\tau\int_0^\infty x g(x,s)\,dx\,ds\Big\} \\
&= C_2(T) + \Big\{\int_0^\infty \Big(1+\frac{\beta}{\alpha}x\Big)u_0(x)\,dx \\
&\quad + \alpha\int_0^\tau\int_0^\infty \Big(1+\frac{\beta}{\alpha}x\Big)g(x,s)\,dx\,ds\Big\} \\
&\leq C_2(T) + \Big\{\|u_0\|_X + \alpha\int_0^\tau \text{ess sup}_{0\leq s\leq\tau\leq T}\int_0^\infty \theta(x)g(x,s)\,dx\,ds\Big\} \\
&\leq C_2(T) + C_0 + \alpha T C_1(T).
\end{aligned}$$

We may, therefore, take R defined by $R := C_2(T) + C_0 + \alpha T C_1(T) \in (0,\infty)$. This proves (P3).

(ii) Proof of (P4). We have to prove that given $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for $\tau \in [0,T]$

$$\sup_{n\geq 1}\Big\{\int_{R_\varepsilon}^\infty u_n(x,\tau)\,dx\Big\} \leq \varepsilon \tag{5.34}$$

Indeed, let us fix $\tau \in [0,T]$, let $\varepsilon > 0$ be arbitrary. Then, for any R_1 we have

$$\int_{R_1}^{\infty} u_n(x,\tau)\,dx = \int_{R_1}^{\infty} \frac{x}{x}\cdot u_n(x,\tau)\,dx \le \frac{1}{R_1}\int_{R_1}^{\infty} x u_n(x,\tau)\,dx. \text{ Using (5.19)} \Rightarrow$$

$$\le \frac{1}{R_1}\Big\{\int_{R_1}^{\infty} x u_0(x)\,dx + \alpha\int_0^{\tau}\int_{R_1}^{\infty} x g(x,s)\,dx\,ds\Big\}$$

$$= \frac{1}{R_1}\Big\{\frac{\alpha}{\beta}\int_{R_1}^{\infty}\Big(1+\frac{\beta}{\alpha}x\Big)u_0(x)\,dx + \frac{\alpha^2}{\beta}\int_0^{\tau}\int_{R_1}^{\infty}\Big(1+\frac{\beta}{\alpha}x\Big)g(x,s)\,dx\,ds\Big\}$$

$$- \frac{1}{R_1}\Big\{\frac{\alpha}{\beta}\int_{R_1}^{\infty} u_0(x)\,dx + \frac{\alpha^2}{\beta}\int_0^{\tau}\int_{R_1}^{\infty} g(x,s)\,dx\,ds\Big\}$$

$$\le \frac{\alpha}{\beta R_1}\int_0^{\infty}\theta(x)u_0(x)\,dx + \frac{\alpha^2}{\beta R_1}\int_0^{\tau}\int_0^{\infty}\theta(x)g(x,s)\,dx\,ds$$

$$\le \frac{\alpha}{\beta R_1}C_0 + \frac{\alpha^2}{\beta R_1}TC_1(T) = \frac{\alpha}{\beta R_1}\big(C_0+\alpha TC_1(T)\big). \tag{5.35}$$

That is,

$$\int_{R_1}^{\infty} u_n(x,\tau)\,dx \le \frac{\alpha\big(C_0+\alpha TC_1(T)\big)}{\beta R_1},\ R_1\in(0,\infty).$$

Therefore, we may choose $R_\varepsilon = R_1$ large enough such that

$$\int_{R_\varepsilon}^{\infty} u_n(x,\tau)\,dx \le \frac{\varepsilon}{2},\ \forall n\ge 1,\ \forall \varepsilon > 0. \tag{5.36}$$

This is for all $n \ge 1$, so (P4) is established.

(iii) Proof of (P5). To prove (P5) we consider $\delta \in (0,\infty)$ and let A be a measurable subset of $\mathbb{R}_+$ having $\mu(A) \le \delta$. Choose $\varepsilon > 0$ such that for all $\tau \in [0,T]$ and $n \ge 1$, (5.36) holds. Then

$$\int_A u_n(x,\tau)\,dx = \int \chi_{A\cap[0,R_\varepsilon]}(x)u_n(x,\tau)\,dx + \int \chi_{A\cap[R_\varepsilon,\infty)}u_n(x,\tau)\,dx$$

$$= \int_0^{R_\varepsilon}\chi_A(x)u_n(x,\tau)\,dx + \int_{R_\varepsilon}^{\infty}\chi_A(x)u_n(x,\tau)\,dx$$

$$\le \sup_{A\subset\mathbb{R}_+}\int_0^{R_\varepsilon}\chi_A(x)u_n(x,\tau)\,dx + \frac{\varepsilon}{2} \quad \text{(by (5.36) for } R_\varepsilon \text{ large enough)}$$

$$= q^n_{\infty,\delta}(\tau) + \frac{\varepsilon}{2} \le C_6(R_\varepsilon,T)\big(q^n_{\infty,\delta}(0) + \delta\alpha TC_1(T)\big) + \frac{\varepsilon}{2}$$

using (5.27), for R_ε large enough. Therefore, for any $\varepsilon > 0$, as u_0 is integrable, it follows by absolute continuity that there exists $\delta_\varepsilon > 0$ so that

$$q^n_{\infty,\delta}(0) + \delta_\varepsilon TC_1(T)\alpha < \frac{\varepsilon}{2C_6(R_\varepsilon,T)} \text{ whenever } \mu(A) < \delta_\varepsilon, \text{ for all } n \ge 1$$

and for R_ε large enough. Therefore, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if $\mu(A) \leq \delta_\varepsilon \Rightarrow$

$$\sup_{n\geq 1} \int_A u_n(x,\tau)\,dx < \varepsilon$$

and (P5) is proved. By (5.33), (5.34) and (P5) and using Dunford-Pettis theorem 2.5, $\mathcal{P} = \{u_n(\tau) : n \geq 1\}$ is a weakly relatively compact subset in $L^1(0,\infty)$. □

Proof of (P2). Choose $\varepsilon > 0$. Taking into account that the following relation

$$\int_{R_1}^{\infty} u_n(x,\tau)\,dx \leq \frac{\alpha(C_0 + \alpha T C_1(T))}{\beta R_1} \tag{5.37}$$

holds uniformly with respect to $\tau \in [0,T]$ and $n \geq 1$, it follows that for R_1 sufficiently large there exists an $a_\varepsilon \geq 1$ s.t.

$$\int_{a_\varepsilon}^{\infty} u_n(x,\tau)\,dx \leq \frac{\varepsilon}{4},\ \forall n \geq 1,\ \forall \tau \in [0,T] \tag{5.38}$$

Let us prove that $\mathcal{P} = \{u_n(\tau) : n \geq 1\}$ is equicontinuous with respect to the strong topology of $L^1(0,\infty)$, i.e.

$$\int_0^{\infty} \left|u_n(x,\tau) - u_n(x,s)\right| dx \leq \varepsilon,\ \forall n \geq 1,$$

whenever s and τ are sufficiently close. Let $\tau \in [0,T]$ and $s \in [0,\tau]$. Using (5.38) we have

$$\begin{aligned}
&\int_0^{\infty} \left|u_n(x,\tau) - u_n(x,s)\right| dx \\
&\quad = \int_0^{a_\varepsilon} \left|u_n(x,\tau) - u_n(x,s)\right| dx + \int_{a_\varepsilon}^{\infty} \left|u_n(x,\tau) - u_n(x,s)\right| dx \\
&\quad \leq \int_0^{a_\varepsilon} \left|u_n(x,\tau) - u_n(x,s)\right| dx + \int_{a_\varepsilon}^{\infty} u_n(x,\tau)\,dx + \int_{a_\varepsilon}^{\infty} u_n(x,s)\,dx \\
&\quad \leq \frac{\varepsilon}{2} + \int_0^{a_\varepsilon} \left|u_n(x,\tau) - u_n(x,s)\right| dx
\end{aligned}$$

Using (5.30),

$$\int_0^{a_\varepsilon} \left|u_n(x,\tau) - u_n(x,s)\right| dx \leq C_7(a,T)(\tau - s)^{\frac{1}{2}} \leq \frac{\varepsilon}{2} \qquad \text{provided that}$$

$$|\tau - s| \leq \delta(\varepsilon,T) := \frac{\varepsilon^2}{4[C_7(a,T)]^2} \tag{5.39}$$

Therefore,

$$\int_0^\infty |u_n(x,\tau) - u_n(x,s)|\, dx \leq \varepsilon,\ \forall n \geq 1 \tag{5.40}$$

and provided that (5.39) holds. Therefore, $\mathcal{P} = \{u_n(\tau) : n \geq 1\}$ is equicontinuous with respect to the strong topology of $L^1(0,\infty)$ which means $\mathcal{P}$ is equicontinuous with respect to the weak topology of $L^1(0,\infty)$. Thus, (P2) hold, which means the set $\mathcal{P}$ is a weakly equicontinuous in $L^1(0,\infty)$. Using the variant of Arzela-Ascoli's Theorem 2.4 it will result that $\mathcal{P}$ is relatively sequentially compact set in $C([0,T]; L^1(0,\infty)_w)$. Thus, Proposition 5.1 is proved. □

Proof of Theorem 5.1. From Proposition 5.1 above and using Arzela-Ascoli's Theorem 2.3 in Chapter 2 there exists a subsequence u_{n_k} of u_n and a function $u \in C([0,+\infty); L^1(0,\infty)_w)$ such that for each $T \in (0,\infty)$ we have: $u_{n_k} \to u$ as $n_k \to \infty$ in $C([0,T]; L^1(0,\infty)_w)$. Since $u(\cdot,\tau)$ is a weak limit of non-negative functions u_{n_k} we may imply that for a.e. $x \in (0,\infty) : u(\cdot,\tau) \geq 0$, for every $\tau \in \mathbb{R}_+$ or $u(\cdot,\tau) \geq 0$, $\forall \tau \in \mathbb{R}_+$.

Moreover, we'll prove that:

$$u \in C([0,\infty); L^1(0,\infty)) \tag{5.41}$$

This means that we have to prove that: for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. for any $(\tau,s) \in \mathbb{R}^2_+$ with $|\tau - s| < \delta$ it results: $\|u(\tau) - u(s)\|_{L^1(0,\infty)} \leq \varepsilon$.
Proof of (5.41). Let $(\tau,s) \in \mathbb{R}^2_+$ and $\varepsilon > 0$ arbitrary. Since $u_{n_k} \rightharpoonup u$ weakly in $L^1(0,\infty)$ with respect to $\tau \in [0,T]$ then $\{u_{n_k}(\tau) - u_{n_k}(s)\}_{k\geq 1}$ converges weakly to $u(\tau) - u(s)$ in $L^1(0,\infty)$. Since $(P2)$ holds then according to (5.40) we have

$$\int_0^\infty |u_n(x,\tau) - u_n(x,s)|\, dx < \varepsilon \tag{5.42}$$

holds uniformly with respect to $n \geq 1$, provided that (5.39) holds. Moreover, the inequality (5.42) holds for any subsequence u_{n_k} of u_n, uniformly with respect to n_k and then we may invert the $\lim_{n_k\to\infty}$ with $\int_0^\infty$ and we get

$$\int_0^\infty \lim_{n_k\to\infty} |u_{n_k}(x,\tau) - u_{n_k}(x,s)|\, ds = \lim_{n_k\to\infty} \int_0^\infty |u_{n_k}(x,\tau) - u_{n_k}(x,s)|\, dx < \varepsilon$$

as long as $(\tau,s) \in \mathbb{R}^2_+$ fulfills (5.39).

Since $\lim_{n_k\to\infty} \big(u_{n_k}(x,\tau) - u_{n_k}(x,s)\big) = u(x,\tau) - u(x,s)$, we have $\int_0^\infty |u(x,\tau) - u(x,s)|\, dx < \varepsilon$, so $\|u(\tau) - u(s)\|_{L^1(0,\infty)} \leq \varepsilon$ as long as $(\tau,s) \in \mathbb{R}^2_+$ satisfies (5.39).

Therefore, $u \in C([0,\infty); L^1(0,\infty))$.

Next, let $T \in (0,\infty)$, $a \in (0,\infty)$ and consider any $R > a$. For $n \geq R$, it follows from the definition of $\theta_n(x)$ that

$$\theta_n(x) = \theta(x),\ \forall x \in [a,R] \subset [0,n].$$

Moreover, $\theta(x)\chi_{[a,R]}(x) \in L^\infty(0,\infty)$, for all $x \in [0,\infty)$, and since $u_{n_k} \rightharpoonup u$ in $L^1(0,\infty)$, it follows from Lemma 5.4 that:

$$\int_0^T \Big(\int_a^R \theta(x) u_{n_k}(x,s)\, dx \Big)^2 ds \le \int_0^T \Big(\int_a^\infty \theta(x) u_{n_k}(x,s)\, dx \Big)^2 ds \le \frac{2}{a\beta}(C_0 + \alpha T C_1(T)) \tag{5.43}$$

Since

$$\theta(x)\chi_{[a,R]}(x) u_{n_k}(x,\cdot) \to \theta(x)\chi_{[a,R]}(x) u(x,\cdot)$$

in $L^1(0,\infty)$ a.e., by the Lebesgue Dominated Convergence Theorem 2.6 in Chapter 2 we get

$$\lim_{n_k \to \infty} \int_0^T \Big(\int_a^R \theta(x) u_{n_k}(x,s)\, dx \Big)^2 ds = \int_0^T \Big(\int_a^R \theta(x) u(x,s)\, dx \Big)^2 ds$$

$$\text{(using (5.43))} \Rightarrow \int_0^T \Big(\int_a^R \theta(x) u(x,s)\, dx \Big)^2 ds \le \frac{2}{a\beta}(C_0 + \alpha T C_1(T))$$

As $R > a$ is arbitrary we get

$$\int_0^T \Big(\int_a^\infty \theta(x) u(x,s)\, dx \Big)^2 ds \le \frac{2}{a\beta}(C_0 + \alpha T C_1(T))$$

The equality (5.18) also holds for $L(x) = \theta(x)\chi_{[0,a]}(x)$ and $s \equiv 0$, and therefore we get:

$$\begin{aligned} \int_0^a \theta(x) u_n(x,\tau)\, dx \le \int_0^a u_0(x)\theta(x)\, dx + \alpha \int_0^\tau \int_0^a g(x,s)\theta(x)\, dx\, ds \\ + \frac{\alpha}{2} \int_0^\tau \int_0^a \int_0^a \phi_n(x,y) u_n(x,s) u_n(y,s) \times \\ \times \big[\theta(x+y) - \theta(x) - \theta(y)\big]\, dy\, dx\, ds \le C_0 + \alpha T C_1(T) \Rightarrow \end{aligned}$$

$$\int_0^a \theta(x)\chi_{[0,a]}(x) u_n(x,\tau)\, dx \le C_0 + \alpha T C_1(T)$$

Since $\theta(x)\chi_{[0,a]}(x) \in L^\infty(0,\infty)$, for any $x \in L^1(0,\infty)$ and $u_n \rightharpoonup u$ in $L^1(0,\infty)$ then by Theorem 2.2 of convergence it will result:

$$\int_0^a \theta(x) u_n(x,\tau)\, dx \to \int_0^a \theta(x) u(x,\tau)\, dx \qquad \text{as } n \to \infty$$

Thus

$$\int_0^a \theta(x) u(x,\tau)\, dx \le C_0 + \alpha T C_1(T) \Rightarrow$$

$$\int_0^T \Big(\int_0^a \theta(x) u(x,\tau)\, dx \Big)^2 d\tau \le T\big(C_0 + \alpha T C_1(T)\big)^2$$

Thus using the well-known inequality $(a+b)^2 \le 2(a^2+b^2)$ for any $a, b \ge 0$ and the above estimate, we have:

$$\begin{aligned}
&\int_0^T \Big(\int_0^\infty \theta(x)u(x,s)\,dx \Big)^2 ds \\
&\quad = \int_0^T \left\{ \int_0^a \theta(x)u(x,s)\,dx + \int_a^\infty \theta(x)u(x,s)\,dx \right\}^2 ds \\
&\quad \le \int_0^T \left\{ 2\Big(\int_0^a \theta(x)u(x,s)\,dx \Big)^2 + 2\Big(\int_a^\infty \theta(x)u(x,s)\,dx \Big)^2 \right\} ds \\
&\quad = 2T\big(C_0 + \alpha T C_1(T)\big)^2 + \frac{2}{a\beta}(C_0 + \alpha T C_1(T)) \\
&\quad = 2T[C_2(T)]^2 + \frac{2}{a\beta}C_2(T) =: C_{14}(a,T) > 0
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^T \Big(\int_0^\infty \int_0^\infty \phi(x,y)u(x,s)u(y,s)\,dy\,dx \Big) ds \\
&\qquad \le \int_0^T \Big(\int_0^\infty \theta(x)u(x,s)\,dx \Big)^2 ds \le C_{14}(a,T)
\end{aligned} \tag{5.44}$$

From the above relation we may conclude that (5.11) in Definition 5.1 is satisfied. Moreover,

$$\|u(\tau)\|_X = \int_0^\infty \theta(x)u(x,\tau)\,dx \le 2(C_0 + \alpha T C_1(T)) = 2C_2(T) =: C_{15}(T) > 0.$$

Taking $\sup_{\tau\in[0,T]}$, it follows that $\sup_{\tau\in[0,T]} \|u(\tau)\|_X \le C_{15}(T)$. Therefore, (a) in Definition 5.1 is satisfied, so $u \in L^\infty(0,\tau;X)$.

Next, we have to prove that (5.13) holds. Relation (5.18) holds for $L(x) = \chi_{[0,R]}(x)$ then (5.18) becomes

$$\int_0^R x u_n(x,\tau)\,dx \le \int_0^R x u_0(x)\,dx + \alpha \int_0^\tau \int_0^R x g(x,\sigma)\,dx\,d\sigma$$

and since for every $x \in (0,\infty)$, $L(x) \in L^\infty(0,\infty)$ and $u_n \rightharpoonup u$ in $L^1(0,\infty)$ then $\int_0^R x u_n(x,\tau)\,dx \to \int_0^R x u(x,\tau)\,dx$, as $n \to \infty$ (using Lebesgue dominated convergence theorem 2.6). Therefore,

$$\int_0^R x u(x,\tau)\,dx \le \int_0^R x u_0(x)\,dx + \alpha \int_0^\tau \int_0^R x g(x,\sigma)\,dx\,d\sigma$$

Since, in particular, relation above holds for R arbitrary then

$$\int_0^\infty xu(x,\tau)\,dx \le \int_0^\infty xu_0(x)\,dx + \alpha\int_0^\tau\int_0^\infty xg(x,\sigma)\,dx\,d\sigma$$

and relation (5.13) is satisfied.

Let us prove now that the function u is indeed a solution to the problem (5.4), (5.5) in the sense of Definition 5.1. For this purpose, let us consider $a \in (1,\infty)$. For $n \ge 1$ and $s \in (0,\tau)$ let $Q \in L^\infty(0,\infty)$ be arbitrary with $\|Q\|_{L^\infty} \le 1$ and $\widetilde{Q}(x,y) = \alpha[Q(x+y) - Q(x) - Q(y)]$. We also define the operators W_i and W_i^n, $n \ge 1$, for i=1, 2, 3 by:

$$W_1^n(a,s) = \int_0^a\int_0^a \phi_n(x,y)\widetilde{Q}(x,y)u_n(x,s)u_n(y,s)\,dx\,dy$$

$$W_1(a,s) = \int_0^a\int_0^a \phi(x,y)\widetilde{Q}(x,y)u(x,s)u(y,s)\,dx\,dy$$

$$W_2^n(a,s) = 2\int_0^a\int_a^\infty \phi_n(x,y)\widetilde{Q}(x,y)u_n(x,s)u_n(y,s)\,dx\,dy$$

$$W_2(a,s) = 2\int_0^a\int_a^\infty \phi(x,y)\widetilde{Q}(x,y)u(x,s)u(y,s)\,dx\,dy$$

$$W_3^n(a,s) = \int_a^\infty\int_a^\infty \phi_n(x,y)\widetilde{Q}(x,y)u_n(x,s)u_n(y,s)\,dx\,dy$$

$$W_3(a,s) = \int_a^\infty\int_a^\infty \phi(x,y)\widetilde{Q}(x,y)u(x,s)u(y,s)\,dx\,dy$$

For $n \ge a$ we have $\phi_n(x,y) \equiv \phi(x,y)$, $\forall (x,y) \in [0,a]\times[0,a] \subseteq [0,n]\times[0,n]$. Since $u_n(s) \rightharpoonup u(s)$ in $L^1(0,\infty)$, $\forall s \in [0,\tau]$ using Theorem 2.2 of weak convergence, in Chapter 2 it will result that

$$\forall \xi \in L^\infty(0,\infty),\ \int_0^\infty \xi(x)u_n(x,\tau)\,dx \to \int_0^\infty \xi(x)u(x,\tau)\,dx \qquad \text{as } n\to\infty.$$

Let us define, for $u \in X^+$ and $x \in [0,a]$, the operator h by

$$h(u)(x,s) = \int_0^a \widetilde{Q}(x,y)\phi_n(x,y)u(y,s)\,dy$$

With the above notation it is clear that

$$W_1^n(a,s) = \int_0^a u_n(x,s)h(u_n)(x,s)\,dx$$

Since $\widetilde{Q} \in L^\infty(0,\infty)$ and $\phi_n(x,y) = \phi(x,y)$, $\forall (x,y) \in [0,a]\times[0,a]$, $\forall n \ge 1$ then according to the Theorem 2.2 of weak convergence, it will result that $h(u_n)(x,s) \to$

$h(u)(x,s)$, as $n \to \infty$ for a.e. $x \in [0,a]$. Also, by using Hölder's inequality and the definition (5.3) of ϕ, we have

$$\begin{aligned}|h(u_n)(x,s)| &\leq \int_0^a |\widetilde{Q}(x,y)|\phi_n(x,y)u_n(y,s)\,dy \leq \theta(a)\|\widetilde{Q}\|_{L^\infty}\int_0^a \theta(y)u_n(y,s)\,dy \\ &< 3\alpha\Big(1+\frac{\beta}{\alpha}a\Big)\int_0^a \theta(y)u_n(y,s)\,dy\end{aligned}$$

Since $|\widetilde{Q}(x,y)| \leq \alpha\Big(|Q(x+y)|+|Q(x)|+|Q(y)|\Big) \leq 3\alpha \Rightarrow$

$$\begin{aligned}|h(u_n)(x,s)| &< 3\alpha\Big(1+\frac{\beta}{\alpha}a\Big)\|u_n(\tau)\|_X < 3(\alpha+a\beta)\sup_{0\leq\tau\leq T}\|u_n(\tau)\|_X \\ &< 3(\alpha+a\beta)C_{15}(T)\end{aligned}$$

Therefore,

$$|h(u_n)(x,s)| < 3(\alpha+a\beta)C_{15}(T), \text{ for a.e. } x \in [0,a]. \tag{5.45}$$

In a similar manner, it will result that $|h(u)(x,s)| \leq 3(\alpha+a\beta)C_{15}(T)$, for a.e. $x \in [0,a]$. Therefore, $h(u_n)$ and $h(u)$ belong both to $L^\infty(0,a)$ (as they are bounded functions, for a.e. $x \in [0,a]$). It now follows by

$$h(u_n)(x,\cdot) \to h(u)(x,\cdot) \text{ as } n \to \infty, \text{ for a.e. } x \in [0,a] \tag{5.46}$$

and Egoroff's Theorem 2.7 that $h(u_n)$ also converges almost uniformly to $h(u)$ on $[0,a]$, or

$$h(u_n) \to h(u) \text{ as } n \to \infty \text{ almost uniformly on } [0,a]. \tag{5.47}$$

This means, by Definition 2.8, Chapter 2 that: given $\delta > 0$ there exists a set $A \subseteq [0,a]$ such that the Lebesgue measure of A: $\mu(A) < \delta$ and

$$h(u_n) \to h(u) \text{ uniformly on } [0,a] \setminus A. \tag{5.48}$$

Let $\varepsilon > 0$ be arbitrary. Furthermore, choose $\varepsilon_1 = \frac{\varepsilon}{2R}$, where R was obtained in the property $(P3)$. Therefore, since (5.48) holds then using Definition 2.7 in Chapter 2, there exists an integer $N_\varepsilon \in \mathbb{N}$ such that $\sup\{|h(u_n)(x)-h(u)(x)| : x \in [0,a]\setminus A\} < \frac{\varepsilon}{2R}$, for any $n \geq N_\varepsilon$. We have

$$\begin{aligned}\|h(u_n)-h(u)\|_{L^\infty([0,a]\setminus A)} &= \inf\{|h(u_n)(x,\cdot)-h(u)(x,\cdot)|, \text{ a.e. for } x \in [0,a]\setminus A\} \\ &\leq \inf\{|h(u_n)(x,\cdot)-h(u)(x,\cdot)|, \text{ for } x \in [0,a]\setminus A\} \\ &= \sup\{|h(u_n)(x,\cdot)-h(u)(x,\cdot)| : x \in [0,a]\setminus A\} < \frac{\varepsilon}{2R},\end{aligned} \tag{5.49}$$

for any $n \geq N_\varepsilon$. Therefore, we have

$$\|h(u_n) - h(u)\|_{L^\infty([0,a]\setminus A)} < \frac{\varepsilon}{2R}, \text{ for any } n \geq N_\varepsilon. \tag{5.50}$$

or

$$h(u_n) \to h(u) \text{ in } L^\infty([0,a] \setminus A), \text{ as } n \to \infty \tag{5.51}$$

Moreover, by property $(P3)$ we have

$$\int_{[0,a]\setminus A} u_n(x,s)\,dx < \int_{[0,a]\setminus A} \theta(x)u_n(x,s)\,dx < R. \tag{5.52}$$

with R defined in $(P3)$.

Furthermore, let $\varepsilon_2 = \frac{\varepsilon}{12R(\alpha+a\beta)C_{15}(T)} > 0$. By Dunford-Pettis Theorem 2.5, since $u_n \rightharpoonup u$ in $L^1(0,\infty)$ it will result that there exists $\delta > 0$ such that

$$\int_E u_n(x,s)\,dx < \varepsilon_2, \text{ whenever } \mu(E) \leq \delta \tag{5.53}$$

Now we use Hölder's inequality and (5.45), (5.50), (5.52), (5.53) with $E = A$ and we have

$$\begin{aligned}
&\left| \int_0^a \Big\{h(u_n)(x,s) - h(u)(x,s)\Big\} u_n(x,s)\,dx \right| \\
&\leq \int_{[0,a]\setminus A} \big|h(u_n)(x,s) - h(u)(x,s)\big| u_n(x,s)\,dx \\
&+ \int_A \big|h(u_n)(x,s) - h(u)(x,s)\big| u_n(x,s)\,dx \\
&\leq \Big(\int_{[0,a]\setminus A} u_n(x,s)\,dx \Big) \|h(u_n) - h(u)\|_{L^\infty([0,a]\setminus A)} \\
&+ \Big(\int_A u_n(x,s)\,dx \Big) \|h(u_n) - h(u)\|_{L^\infty(A)} \\
&< R\|h(u_n) - h(u)\|_{L^\infty([0,a]\setminus A)} + \varepsilon_2 \cdot 6(\alpha + a\beta C_{15}(T)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for any } n \geq N_\varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrary, we have

$$\left| \int_0^a \Big\{h(u_n)(x,s) - h(u)(x,s)\Big\} u_n(x,s)\,dx \right| \to 0 \text{ as } n \to \infty$$

Since $h(u) \in L^\infty(0,a)$ and $u_n \rightharpoonup u$ in $L^1(0,\infty)$ as $n \to \infty$ then by Theorem 2.2, Chapter 2 it will result

$$\int_0^a h(u)(x,s)u_n(x,s)\,dx \to \int_0^a h(u)(x,s)u(x,s)\,dx \text{ as } n \to \infty, \text{ or}$$

$$\left| \int_0^a h(u)(x,s)\big[u_n(x,s) - u(x,s)\big]\,dx \right| \to 0 \qquad \text{as } n \to \infty$$

Thus

$$\begin{aligned}
\left|W_1^n(a,s) - W_1(a,s)\right| &= \Big| \int_0^a u_n(x,s)h(u_n)(x,s)\,dx - \int_0^a u(x,s)h(u)(x,s)\,dx \Big| \\
&= \Bigg| \Big(\int_0^a u_n(x,s)h(u_n)(x,s)\,dx - \int_0^a h(u)(x,s)u_n(x,s)\,dx \Big) \\
&+ \int_0^a \Big(u_n(x,s)h(u)(x,s) - u(x,s)h(u)(x,s) \Big)\,dx \Bigg| \\
&< \left| \int_0^a \big[h(u_n)(x,s) - h(u)(x,s)\big]u_n(x,s)\,dx \right| + \left| \int_0^a h(u)(x,s)\big[u_n(x,s) - u(x,s)\big]\,dx \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \qquad (5.54)
\end{aligned}$$

As ε was chosen arbitrary then $\left|W_1^n(a,s) - W_1(a,s)\right| \to 0$ as $n \to \infty$. Therefore, $W_1^n(a,s) \to W_1(a,s)$ in $L^1(0,a)$ as $n \to \infty$ and $\lim_{n\to\infty} W_1^n(a,s) = W_1(a,s)$

Using (5.25) and since $\phi_n(x,y) \equiv \phi(x,y)$ on $[0,a] \times [0,a]$ we have

$$\begin{aligned}
|W_1^n(a,s)| &\le \int_0^a \int_0^a \phi_n(x,y)|\widetilde{Q}(x,y)|u_n(x,s)u_n(y,s)\,dx\,dy \\
&\le \int_0^a \int_0^a \phi(x,y)\|\widetilde{Q}\|_\infty u_n(x,s)u_n(y,s)\,dx\,dy \\
&< 6\widetilde{\phi}_0(a) \int_0^a \int_0^a u_n(x,s)u_n(y,s)\,dx\,dy \\
&= 6\widetilde{\phi}_0(a)\Big(\int_0^a u_n(x,s)\,dx \Big)^2 < 6\widetilde{\phi}_0(a)[C_2(T)]^2 \Rightarrow \\
|W_1^n(a,s)| &< 6\widetilde{\phi}_0(a)[C_2(T)]^2
\end{aligned}$$

Since $\lim_{n\to\infty} W_1^n(a,s) = W_1(a,s)$ then by the Lebesgue Dominated Convergence Theorem 2.6,

$$\lim_{n\to\infty} \int_0^\tau W_1^n(a,s)\,ds = \int_0^\tau W_1(a,s)\,ds \qquad (5.55)$$

We have

$$W_2^n(a,s)+W_3^n(a,s)=2\int_0^a\int_a^\infty \phi_n(x,y)\widetilde{Q}(x,y)u_n(x,s)u_n(y,s)\,dx\,dy$$
$$+\int_a^\infty\int_a^\infty \phi_n(x,y)\widetilde{Q}(x,y)u_n(x,s)u_n(y,s)\,dx\,dy$$
$$=2\int_0^\infty\int_a^\infty \phi_n(x,y)\widetilde{Q}(x,y)u_n(x,s)u_n(y,s)\,dx\,dy$$
$$-\int_a^\infty\int_a^\infty \phi_n(x,y)\widetilde{Q}(x,y)u_n(x,s)u_n(y,s)\,dx\,dy$$

Therefore,

$$|W_2^n(a,s)+W_3^n(a,s)|<9\alpha\Big\{\int_0^\infty\int_a^\infty \phi_n(x,y)u_n(x,s)u_n(y,s)\,dx\,dy$$
$$+\int_a^\infty\int_a^\infty \phi_n(x,y)u_n(x,s)u_n(y,s)\,dx\,dy\Big\}$$
$$<9\alpha\int_0^\infty\int_a^\infty \theta_n(x)\theta_n(y)u_n(x,s)u_n(y,s)\,dx\,dy$$

Using (5.23), (5.26) and Hölder's inequatlity we get

$$\int_0^\tau |W_2^n(a,s)+W_3^n(a,s)|\,ds$$
$$<9\alpha\int_0^\tau\Big(\int_0^\infty\int_a^\infty \theta_n(x)\theta_n(y)u_n(x,s)u_n(y,s)\,dx\,dy\Big)\,ds$$
$$=9\alpha\int_0^\tau\Big(\int_0^\infty \theta_n(x)u_n(x,s)\,dx\Big)\Big(\int_a^\infty \theta_n(x)u_n(x,s)\,dx\Big)\,ds$$
$$\le 9\alpha\Big\{\int_0^\tau\Big(\int_0^\infty \theta_n(x)u_n(x,s)\,dx\Big)^2 ds\Big\}^{\frac12}\Big\{\int_0^\tau\Big(\int_a^\infty \theta_n(x)u_n(x,s)\,dx\Big)^2 ds\Big\}^{\frac12}$$
$$\le 9\alpha\sqrt{\frac{2}{\alpha}C_2(T)}\sqrt{\frac{2}{a\alpha}C_2(T)}=\frac{18C_2(T)}{\sqrt{a}}$$

Therefore, we have

$$\int_0^\tau |W_2^n(a,s)+W_3^n(a,s)|\,ds<\frac{18C_2(T)}{\sqrt{a}}.$$

Then, using a similar procedure with ϕ instead of with ϕ_n and the following inequalities:

$$\int_0^\tau\Big(\int_a^\infty \theta(x)u(x,s)\,dx\Big)^2 ds\le\int_0^T\Big(\int_a^\infty \theta(x)u(x,s)\,dx\Big)^2 ds\le\frac{2}{a\beta}C_2(T)$$
$$\int_0^\tau\Big(\int_0^\infty \theta(x)u(x,s)\,dx\Big)^2 ds\le\int_0^T\Big(\int_0^\infty \theta(x)u(x,s)\,dx\Big)^2 ds\le C_{14}(a,T)$$

where $C_{14}(a,T) = 2T(C_2(T))^2 + \frac{2}{a\beta}C_2(T)$ we get

$$\int_0^\tau \left|W_2(a,s) + W_3(a,s)\right| ds \leq 9\alpha\sqrt{\frac{2}{a\beta}C_2(T)C_{14}(a,T)}$$

By the above relations, from the Lebesgue Dominated Convergence Theorem 2.6 and using (5.55) it will result

$$\lim_{n\to\infty}\sup_{0\leq\tau\leq T}\left|\int_0^\tau \Big[\big(W_1^n(a,s) + W_2^n(a,s) + W_3^n(a,s)\big) - \big(W_1(a,s) + W_2(a,s) + W_3(a,s)\big)\Big] ds\right| \leq \frac{18C_2(T)}{\sqrt{a}} + 9\alpha\sqrt{\frac{2}{a\beta}C_2(T)C_{14}(a,T)} \tag{5.56}$$

We have

$$W_1^n(a,s) + W_2^n(a,s) + W_3^n(a,s) = \int_0^\infty\int_0^\infty \phi_n(x,y)\widetilde{Q}(x,y)u_n(x,s)u_n(y,s)\,dx\,dy$$
$$W_1(a,s) + W_2(a,s) + W_3(a,s) = \int_0^\infty\int_0^\infty \phi(x,y)\widetilde{Q}(x,y)u(x,s)u(y,s)\,dx\,dy$$

At this point notice that $W_1^n(a,s) + W_2^n(a,s) + W_3^n(a,s)$ and $W_1(a,s) + W_2(a,s) + W_3(a,s)$ do not depend on $a \in (1,\infty)$, which means the above inequality (5.56) holds for every $a \in (1,\infty)$. The above inequality being valid for every $a \in (1,\infty)$ we finally obtain

$$\lim_{n\to\infty}\int_0^\tau\int_0^\infty\int_0^\infty \widetilde{Q}(x,y)\phi_n(x,y)u_n(x,s)u_n(y,s)\,dx\,dy\,ds = \int_0^\tau\int_0^\infty\int_0^\infty \widetilde{Q}(x,y)\phi(x,y)u(x,s)u(y,s)\,dx\,dy\,ds \tag{5.57}$$

Since $Q \in L^\infty(0,\infty)$ and $u_n \rightharpoonup u$ in $L^1(0,\infty)$ then

$$\lim_{n\to\infty}\int_0^\infty [u_n(x,\tau) - u_0(x)]Q(x)\,dx = \int_0^\infty [u(x,\tau) - u_0(x)]Q(x)\,dx$$

then passing to $\lim_{n\to\infty}$ in Lemma 5.2, (5.18), u satisfies

$$\int_0^\infty Q(x)\big(u(x,\tau) - u_0(x)\big)\,dx = \frac{1}{2}\int_0^\tau\int_0^\infty\int_0^\infty \phi(x,y)\widetilde{Q}(x,y)u(x,s)u(y,s)\,dx\,dy\,ds + \int_0^\tau\int_0^\infty \alpha g(x,s)Q(x)\,dx\,ds$$

Using Fubini's Theorem in the first integral on the right-hand side, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^\tau \int_0^\infty \int_0^\infty \phi(x,y)\widetilde{Q}(x,y)u(x,s)u(y,s)\,dx\,dy\,ds \text{ (by def of } \widetilde{Q}) \\
&= \frac{\alpha}{2}\int_0^\tau \int_0^\infty \int_0^\infty \phi(x,y)Q(x+y)u(x,s)u(y,s)\,dx\,dy\,ds \\
&- \frac{\alpha}{2}\int_0^\tau \int_0^\infty \int_0^\infty \phi(x,y)[Q(x)+Q(y)]u(x,s)u(y,s)\,dx\,dy\,ds \\
&= \int_0^\tau \int_0^\infty Q(x)\int_0^x \frac{\alpha}{2}\phi(x-y,y)u(x-y,s)u(y,s)\,dy\,dx\,ds \\
&- \int_0^\tau \int_0^\infty u(x,s)\int_0^\infty \alpha\phi(x,y)u(y,s)\,dy\,dx\,ds
\end{aligned}$$

Therefore, we get:

$$\begin{aligned}
\int_0^\infty Q(x)[u(x,\tau)-u_0(x)]\,dx &= \int_0^\infty Q(x)\int_0^\tau \left\{\frac{\alpha}{2}\int_0^x \phi(x-y,y)u(x-y,s)u(y,s)\,dy\right. \\
&\left. - u(x,s)\int_0^\infty \alpha\phi(x,y)u(y,s)\,dy\right\} ds\,dx + \alpha\int_0^\tau\int_0^\infty g(x,s)Q(x)\,dx\,ds
\end{aligned}$$

Since the above inequality holds for every $Q \in L^\infty(0,\infty)$ then we have shown that u satisfies equation (5.4), (5.5) which means that u is a solution on $[0,\infty)$ in the sense of Definition 5.1 with $u(0) = u_0$ and statisfies

$$\int_0^\infty xu(x,\tau)\,dx \le \int_0^\infty xu_0(x)\,dx + \alpha\int_0^\tau\int_0^\infty xg(x,s)\,dx\,ds$$

and this completes the proof of theorem 5.1. □

Note. The proof of the convergence of the operator W_1^n to W_1 presented above can be stated as a lemma regarding the continuity property of some bilinear integral operator with respect to the weak topology of $L^1 \times L^1$.

Lemma 5.7. Consider $0 < a \le b < \infty$ and $H \in L^\infty((0,a)\times(0,b))$. We define a mapping W on $L^1(0,a)\times L^1(0,b)$ by $W(\phi,\psi) = \int_0^a\int_0^b H(x,y)\phi(x,s)\psi(y,s)\,dx\,dy$. If $\{\phi_n\}_{n\ge1}$ is a sequence in $L^1(0,a)$ converging weakly to ϕ in $L^1(0,a)$ and $\{\psi_n\}_{n\ge1}$ is a sequence in $L^1(0,b)$ converging weakly to ψ in $L^1(0,b)$ then there holds

$$\lim_{n\to\infty} W(\phi_n,\psi_n) = W(\phi,\psi)$$

Conclusion.
In this chapter we proved the existence of solutions to the coagulation equation with

particle sources (5.1) with initial condition (5.2) in a suitable Banach space X for product-type coagulation kernel K satisfying (4.3), (4.4). The results are interesting as they generalize previous arguments given by Laurençot in [30] for coagulation equations with fragmentation. Results regarding the global uniqueness of solutions of (5.1) are left for the future work.

Chapter 6

Mass-Conserving Solutions and Gelation

6.1 Gelation in the pure coagulation models

The purpose of this chapter is to investigate the existence of mass-conserving solutions and the occurence of gelation in the pure Smoluchowski coagulation equation describing the evolution of a system of particles which are continuously growing as a result of pairs of particles coming into contact and adhering or bonding to form larger and larger clusters, under various assumptions on the coagulation kernel. Denoting by $c(x,t)$ the density of clusters of size x at time t (or the size distribution function at time t), the continuous Smoluchowski coagulation equation reads:

$$\frac{\partial c}{\partial t}(x,t) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)\,dy - c(x,t)\int_0^\infty K(x,y)c(y,t)\,dy \quad (6.1)$$

$$c(x,0) = c_0(x) \geq 0 \quad (6.2)$$

For pure coagulation equation (without fragmentation and other processes) the coagulation must lead to the dissipation of the total amount of particles at time t which is expressed by the zero moment of solution $c(x,t)$ as:

$$M_0(t) = \int_0^\infty c(x,t)\,dx \quad (6.3)$$

As already shown in [13], the mathematical treatment of equation (6.1) confirms that the dissipation law is valid, which means: $\frac{dM_0(t)}{dt} \leq 0$ or $M_0(t_1) \leq M_0(t_0)$, for any $t_1 \geq t_0 \geq 0$ (the number of particles decreases). Moreover, from the definition of the distribution function c, we conclude that the total mass of particles per unit volume at time t is expressed by the first moment of $c(x,t)$:

$$M_1(t) = \int_0^\infty xc(x,t)\,dx \quad (6.4)$$

The solutions to the coagulation equation (6.1) for which the following equality

$$\int_0^\infty xc(x,t)\,dx = \int_0^\infty xc_0(x)\,dx \text{ or } M_1(t) = M_1(0) \tag{6.5}$$

holds for all $t \geq 0$ are defined as mass-conserving solutions or density-conserving solutions. From a physical point of view, the mass-conservation law is very natural. It is expected that in the coagulation process described by (6.1), where there is neither sink nor source of clusters, the total mass $M_1(t)$ to be constant throughout time evolution, i.e. $M_1(t) = M_1(0)$, for all $t \in \mathbb{R}_+$.

This is indeed what happens at a formal level: We multiply the equation (6.1) by x and assume that all integrals in (6.1) exist. On using the change of variables $x' = x - y$, $y' = y$ in the first right-hand side integral of convolution type and the symmetry property of the coagulation kernel K, the result is $\dfrac{dM_1(t)}{dt} = 0$, for all $t \in \mathbb{R}_+$.

While this is true in some cases (see, e.g. [31], [47]), it is well known that the formal computation just suggested cannot be made rigorous in all situations: For some rate coefficients $K(x,y)$ increasing sufficiently fast with x and y, it has been proved rigorously that $M_1(t)$ cannot be constant for all t and the property $M_1(t) = M_1(0)$, for all $t \in \mathbb{R}_+$, fails to be true. Therefore, there must exist a non-negative time T_{gel} such that $M_1(t) = M_1(0)$, for $t \leq T_{gel}$ but $M_1(t) < M_1(0)$, for $t > T_{gel}$ (see [18, 19, 30, 39]). In the literature this phenomenon is known as the gelation problem and it is interpreted physically as corresponding to the occurence of a dynamic phase transition in the system or by the appearance of an infinite "gel" or "superparticle". In the pure coagulation equation, the "gel" or "gelation time", T_{gel} was defined as the first instance when mass-conservation breaks down, that is:

$$T_{gel} := \inf\left\{t \in \mathbb{R}_+ : M_1(t) < M_1(0)\right\} \in [0, +\infty] \tag{6.6}$$

If the rate of production of particles is very high, a portion of the total mass of the system is rapidly transferred to larger and larger clusters and can be carried to a "macroscopic" cluster with an infinite number of particles (i.e., $x \to \infty$) in a finite time T_{gel}. This new phase, called the "gel" or "superparticle", is not modelled by the equations (6.1) and has a mass proportional to $M_1(0) - M_1(t)$ for $t > T_{gel}$.

This dynamic phase transition has attracted a great deal of interest in recent years, much effort being spent on trying to characterize the coagulation kernels and initial conditions for which gelation occurs as well as the properties of the gelling solutions. The possibility of occurence of gelation for some type of coefficients has been settled negatively in some cases.

In particular, the following have been established rigorously:
(A) If the coagulation kernel $K(x,y)$ satisfies $K(x,y) \leq M$, for some constant $M > 0$ and for all $(x,y) \in \mathbb{R}_+^2$ then all solutions conserve mass (or density) $M_1(t)$ for all time $t \in \mathbb{R}_+$ (see [31]).

(B) If K satisfies $K(x,y) \leq M(x+y+1)$, whenever $(x,y) \in \mathbb{R}_+^2$ for some constant $M > 0$ then all solutions conserve mass $M_1(t)$ for all time $t \in \mathbb{R}_+$ (see [47]).
(C) If $K(x,y) = (xy)^\alpha$, when $\alpha \in [0, 1/2]$ and for all $(x,y) \in \mathbb{R}_+^2$.
This means that no gelation can occur in the above cases.

It is worth to present here an important result regarding the occurence of the gelation time in the coagulation equation:

Corollary 6.1. (i) If $T_{gel} = \infty$, then no gelation occurs, that is the (total) mass, $M_1(t)$, is being conserved or is kept constant for all time $t \geq 0$.
(ii) If $T_{gel} < \infty$ then gelation occurs.

The coagulation kernels of interest are of the product type: $K(x,y) = K_0 xy$ for some positive constant K_0. It was believed for a long time that solutions of Smoluchowski's coagulation equation, with the coagulation kernel $K(x,y) = xy$ and for suitably restricted initial particle distributions $c_0(x)$, exist only for small times. McLeod [32], has shown that if $xc_0(x) = e^{-x}$ and $K(x,y) = xy$, for all $x, y \geq 0$ then there is a mass-conserving solution on some local time interval (see also Drake ([12], p.254-256) for a short presentation of these results). Ernst et al. [18] were the first to demonstrate that the conservation of mass need not hold for a coagulation kernel $K(x,y) \sim (xy)^w$ with $\frac{1}{2} < w \leq 1$ and that (6.1) describes a gelation transition (formation of an infinite cluster after a finite time t_c). In the article, Ernst et al. [18] obtained explicit solutions for the coagulation equation (6.1) with $K(x,y) = xy$ for all times and for a general initial condition u_0 and showed that they are physically meaningful. They also proved that there occurs a phase transition (gelation) at the gel point $t_c = 1$ such that: in the sol phase ($t < t_c = 1$), $M_1(t)$ is constant and in the gel phase ($t > t_c = 1$), the total mass $M_1(t)$ contained in the clusters of finite size (sol particles) is no longer conserved and decreases to zero as time progresses.

Results regarding asymptotic properties of the total mass $M_1(t)$ as $t \to t_c$ and $t \to \infty$ have been obtained for special examples of initial distributions $c_0(x)$. In this sense we present here two of the important results obtained in [18].

Remark1: Corresponding to the small x-behaviour of the initial size distribution, e.g. $u_0(x) \cong \dfrac{Ax^{p-2}}{\Gamma(p)}$ (with $A, p > 0$) for $x \to 0$, the resulting sol mass decreases asymptotically as

$$M_1(t) \cong A(Apt)^{\frac{-p}{(p+1)}} \qquad (\text{as } t \to \infty).$$

Remark2: If the initial size distribution $u_0(x) = 0$, for $x \in (0, x_0)$ then asymptotically

$$M_1(t) \cong (x_0 t)^{-1} \quad . \ (\text{as } t \to \infty)$$

By using the Laplace transform method, Ernst et al.[18] obtained the general solution

of (6.1) and represented it in the form of a series expansion

$$c(x,t) = \frac{e^{-xT}}{x^2 t} \sum_{k=0}^{\infty} \frac{(xt)^k}{k!} \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} (u(z))^k e^{xz} dz$$

where $u(z) = \int_0^\infty x u_0(x) e^{-xz}\, dx$ and $T = \int_0^t M_1(s)\, ds$.

It has recently been shown rigorously (see [39]) for the class of coagulation kernels $K(x,y) = f(x)f(y)$, where $f(x) = \alpha + \beta x$, $\alpha, \beta \geq 0$ that the total mass breaks down at a finite time $T = -\frac{1}{\beta h'(0^+)}$, where $h(x)$ is a completely monotonic function for $x \geq 0$ which can be expressed from Bernstein's theorem 2.8 as $h(x) = \int_0^\infty e^{-xz} f(z) u_0(z)\, dz$. Therefore, for $t < T$ the conservation of mass holds. For T finite and $t > T$ the post-gelation solution of (6.1) has been investigated for two different cases ($\alpha = 0$ and $\alpha \neq 0$), an expression for the total mass being obtained.

The coagulation equation (6.1), for the class of coagulation kernels defined above, may be rewritten in the following form

$$\frac{\partial c}{\partial t}(x,t) = \frac{1}{2} \int_0^x \alpha^2 [1 + \frac{\beta}{\alpha}(x-y)][1 + \frac{\beta}{\alpha} y] c(x-y,t) c(y,t)\, dy$$
$$- c(x,t) \int_0^\infty \alpha^2 (1 + \frac{\beta}{\alpha} x)(1 + \frac{\beta}{\alpha} y) c(y,t)\, dy$$

Let $u(x,t) := \alpha c(x,t)$ we get

$$\frac{\partial u}{\partial t}(x,t) = \frac{\alpha}{2} \int_0^x \phi(x-y,y) u(x-y,t) u(y,t)\, dy - \alpha u(x,t) \int_0^\infty \phi(x,y) u(y,t)\, dy \tag{6.7}$$

$$u(x,0) = \alpha c_0(x) = u_0(x) \geq 0 \qquad \text{a.e}$$

We are interested to determine the large-time behaviour of the total mass $M(t)$ of solutions u of the equation (6.7) defined by: $M(t) = \int_0^\infty x u(x,t)\, dx$, $t \in \mathbb{R}_+$ for the coagulation kernels $\phi(x,y) = (1 + \frac{\beta}{\alpha} x)(1 + \frac{\beta}{\alpha} y)$, $x, y \in \mathbb{R}_+$, with α, β positive constants, $\alpha > 0$. The coagulation kernel K mentioned above is a particular case of the coagulation kernel assumed by Laurençot (see [30]). The following theorem is a consequence of Laurençot's results, regarding the long-time behaviour of the total mass $M(t)$ and illustrates once more the occurence of gelation for the class of coagulation kernels ϕ mentioned above and where the initial distribution u_0 belongs to the space X^+. Here X^+ represents the positive cone of the Banach space $X = L^1(0,\infty; \theta(x)\, dx)$, where $\theta(x) = 1 + \frac{\beta}{\alpha} x$.

Theorem 6.1. (see Laurençot [30], p.250) Assume that ϕ satisfies $\phi(x,y) = \theta(x)\theta(y)$, for $(x,y) \in \mathbb{R}_+^2$ where θ satisfies $\theta(x) = 1 + \frac{\beta}{\alpha} x$, for some positive constants $\alpha > 0, \beta \geq 0$.

(A) Consider $u_0 \in X^+, u_0 \neq 0$ and let u be a solution to (6.7) on $[0,\infty)$ with initial data u_0. Then

$$M(t) := \int_0^\infty xu(x,t)\,dx \leq \frac{\sqrt{2}}{\sqrt{t}}\frac{\sqrt{\alpha}}{\beta}\Big(\int_0^\infty u_0(x)\,dx\Big)^{\frac{1}{2}} \qquad \forall t \in (0,\infty) \tag{6.8}$$

(B) If, in addition, u_0 satisfies

$$I_q := \int_0^\infty x^{-q}u_0(x)\,dx < \infty \qquad \text{for some } q \in (0,\infty)$$

then there holds:

$$M(t) \leq M(0)\min\Big\{1, \Big(\frac{q+(q+2)t/T^*}{2(q+1)}\Big)^{-\frac{q+1}{q+2}}\Big\} \tag{6.9}$$

where

$$T^* = 2M(0)^{-m}\Big(\frac{\alpha}{\beta^2}\Big)(I_q)^{\frac{1}{q+1}} \qquad \text{with } m = \frac{q+2}{q+1}$$

(C) If $u_0 \equiv 0$ on $(0,\delta)$ for some $\delta > 0$ we have

$$M(t) \leq M(0)\min\Big\{1, \Big(\frac{1+t/T^*}{2}\Big)^{-1}\Big\} \qquad \text{where } T^* = \frac{2}{\delta M(0)}\Big(\frac{\alpha}{\beta^2}\Big) \tag{6.10}$$

The key point in the proof of the Theorem 6.1 is the following identity

Lemma 6.1. Let $Q \in L^\infty(0,+\infty)$. For $t \in (0,+\infty)$ and $s \in [0,t)$ there holds

$$\int_0^\infty Q(x)[u(x,t)-u(x,s)]\,dx = \frac{1}{2}\int_s^t\int_0^\infty\int_0^\infty \phi(x,y)\widetilde{Q}(x,y)u(x,\sigma)u(y,\sigma)\,dx\,dy\,d\sigma \tag{6.11}$$

where $\widetilde{Q}$ is defined by $\widetilde{Q}(x,y) = \alpha\big(Q(x+y)-Q(x)-Q(y)\big)$. □

As a consequence of Lemma 6.1 we obtain the following important result:

Lemma 6.2. (see [30], p.268) For $t \in (0,\infty)$ and $s \in [0,t)$ there holds $M(t) \leq M(s)$ (the total mass decreases). Next, let $w : (0,\infty) \to \mathbb{R}_+$ be a non-negative and non-increasing function such that $w(x+y) \leq w(x)+w(y)$, $(x,y) \in (0,\infty)^2$. Then, if u_0 satisfies the integrability property $\int_0^\infty w(x)u_0(x)\,dx < \infty$ then $u(\cdot,t)$ satisfies itself the integrability property $\int_0^\infty w(x)u(x,t)\,dx < \infty$. Moreover, for $t > s \geq 0$ we have

$$\int_0^\infty w(x)u(x,t)\,dx \leq \int_0^\infty w(x)u(x,s)\,dx$$

The proof of the Theorem 6.1 follows closely the lines of the proof given by Laurençot in ([30], Theorem 1.3). In order to prove (B) and (C), Laurençot uses the following important nonlinear integral inequality (see Komornik [29], Theorem 9.1):

Theorem 6.2. Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $\gamma > 0$ and $T > 0$ such that

$$\int_t^\infty E^{\gamma+1}(s)\,ds \leq TE(0)^\gamma E(t) \qquad \forall t \in \mathbb{R}_+$$

Then we have

$$E(t) \leq E(0)\Big(\frac{T+\gamma t}{T+\gamma T}\Big)^{-\frac{1}{\gamma}}$$

From Theorem 6.1 it follows that the temporal decay of the total mass M depends strongly on the amount of clusters of very small size ($x \approx 0$) in the initial distribution u_0. As already noticed in [18], when $K(x,y) = xy$ for some specific initial distribution (for which explicit solutions were available), the smaller the amount of clusters, the faster the decay of the total mass M. Theorem 6.1 comes as an extension of the results presented by Ernst et al. [18].

Theorem 6.1 is also important as it gives an upper bound on the gelation time defined by (6.6). Indeed, it follows from (6.8) that T_{gel} is not greater than the root of

$$\frac{\sqrt{2}}{\sqrt{t}}\frac{\sqrt{\alpha}}{\beta}\Big(\int_0^\infty u_0(x)\,dx\Big)^{\frac{1}{2}} < M(0) \tag{6.12}$$

and therefore it results that an upper bound for T_{gel} would be

$$T_{gel} \leq 2\Big(\frac{\alpha}{\beta}\Big)^2 \frac{\int_0^\infty u_0(x)\,dx}{M(0)^2}$$

6.2 Gelation in the coagulation equation with source terms

In this section we continue our study of the large time behaviour of the total mass $M(t) = \int_0^\infty xc(x,t)\,dx$ of solutions of the coagulation equation with source terms:

$$\frac{\partial c}{\partial t}(x,t) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)\,dy - \int_0^\infty K(x,y)c(x,t)c(y,t)\,dy + g(x,t) \tag{6.13}$$

subject to initial condition

$$c(x,0) = c_0(x) \geq 0 \tag{6.14}$$

and with coagulation kernels of the form $K(x,y) = f(x)f(y)$, where $f(x) = \alpha + \beta x$, for $\alpha > 0, \beta \geq 0$ some positive constants. Let us denote by c a solution to the equation (6.13), (6.14) whose existence has been obtained in Chapter 5. A similar transformation to the one we have already used in the previous section, which is

$$u(x,t) := \alpha c(x,t) \tag{6.15}$$

will lead us to the following coagulation equation with source terms

$$\frac{\partial u}{\partial t}(x,t) = \frac{\alpha}{2}\int_0^x \phi(x-y,y)u(x-y,t)u(y,t)\,dy - \alpha u(x,t)\int_0^\infty \phi(x,y)u(y,t)\,dy + \alpha g(x,t) \tag{6.16}$$

$$u(x,0) = \alpha c_0(x) = u_0(x) \geq 0 \qquad \text{a.e} \tag{6.17}$$

According to the transformation (6.15) u still defines a solution to the equation (6.16), (6.17) which exists as we have just obtained in Chapter 5.

In the following we introduce the (total) number of clusters at time t to be defined by

$$M_0(t) := \int_0^\infty u(x,t)\,dx,\ \forall t \in \mathbb{R}_+$$

We also introduce the (total) mass of clusters per unit volume at time t (or the total density of solution u of the equation (6.16), (6.17)) defined by

$$M_1(t) := \int_0^\infty xu(x,t)\,dx,\ \forall t \in \mathbb{R}_+ \tag{6.18}$$

The transformation (6.15) changes the coagulation kernel K into a new kernel ϕ which obviously, satisfies the assumptions (4.3), (4.4) and has the following form

$$\phi(x,y) = \theta(x)\cdot\theta(y),\ \forall x,y \in \mathbb{R}_+ \qquad \text{where}$$

$$\theta(x) = 1 + \frac{\beta}{\alpha}x,\ \forall \alpha > 0, \beta \geq 0 \tag{6.19}$$

We also assume that the initial distribution u_0 satisfies

$$u_0 \in X^+ \tag{6.20}$$

where the Banach space X was already defined in (5.6) endowed with the norm $\|\cdot\|_X$ defined by (5.7) and the source term g satisfies the hypothesis:

$$g \in L^1(0,\infty;X) \tag{6.21}$$

A physically relevant and mathematically challenging question is to figure out whether the total mass (or the total density) $M_1(t)$ of solutions to the coagulation

equation with sources (6.16) remains constant throughout time evolution, that is if $M_1(t) = M_1(0)$.

As we have already mentioned in the previous section, several works have considered this question for the pure coagulation case (no source terms, see, e.g. [18], [39]). Either formal or explicit solutions have been provided in this case to show that the solution need not conserve mass for all time.

In the following we are concerned with the large-time behaviour of the total density (mass) $M_1(t)$ of the solutions u of (6.16). Our main goal is to extend the results previously obtained for the pure coagulation equation by Laurençot [30] in Theorem 6.1. Using Definition 5.1 in Chapter 5 we obtain the following result:

Lemma 6.3. Let $Q \in L^\infty(0,\infty)$. For $t \in (0,\infty)$ and $s \in [0,t)$ the following identity holds:

$$\int_0^\infty Q(x)\cdot[u(x,t)-u(x,s)]\,dx = \frac{1}{2}\int_s^t\int_0^\infty\int_0^\infty \phi(x,y)\widetilde{Q}(x,y)u(x,\sigma)u(y,\sigma)\,dx\,dy\,d\sigma + \alpha\int_s^t\int_0^\infty Q(x)g(x,\sigma)\,dx\,d\sigma \tag{6.22}$$

where $\widetilde{Q}(x,y)$ is defined by $\widetilde{Q}(x,y) = \alpha\big(Q(x+y)-Q(x)-Q(y)\big)$. □

As a consequence of Lemma 6.3 we obtain the following property regarding the total density M_1.

Lemma 6.4. For $t \in (0,\infty)$ and $s \in [0,t)$ there holds

$$M_1(t) \le M_1(s) + \alpha\int_s^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma \tag{6.23}$$

Let $h : (0,\infty) \to \mathbb{R}_+$ be a non-negative and non-increasing function satisfying the subadditivity property

$$h(x+y) \le h(x)+h(y),\ \forall x,y \in (0,\infty)\times(0,\infty) \tag{6.24}$$

Then, if u_0 and g satisfy the additional integrability properties:

$$\int_0^\infty h(x)u_0(x)\,dx < \infty \ \text{ and } \int_0^\infty h(x)g(x,s)\,dx < \infty \tag{6.25}$$

then $u(\cdot,t)$ also satisfies the integrability property:

$$\int_0^\infty h(x)u(x,t)\,dx < \infty. \tag{6.26}$$

Moreover, for $t > s \ge 0$

$$\int_0^\infty h(x)u(x,t)\,dx \le \int_0^\infty h(x)u(x,s)\,dx + \alpha\int_s^t\int_0^\infty h(x)g(x,\sigma)\,dx\,d\sigma \tag{6.27}$$

Proof Let $L \in (0,\infty)$ and take $Q(x) = x \cdot \chi_{[0,L]}(x)$ in the identity (6.22). It follows that if $(x,y) \in [0,L] \times [0,L] \Rightarrow \widetilde{Q}(x,y) \leq \alpha(x+y-x-y) = 0$, while if $x \geq L$ or $y \geq L \Rightarrow \widetilde{Q}(x,y) \leq \alpha(-Q(x) - Q(y)) \leq 0$. In all cases, we obtain $\widetilde{Q}(x,y) \leq 0$ and thus the identity (6.22) leads to

$$\int_0^L x[u(x,t) - u(x,s)]\,dx \leq \alpha \int_s^t \int_0^L xg(x,\sigma)\,dx\,d\sigma$$

or

$$\int_0^L xu(x,t)\,dx \leq \int_0^L xu(x,s)\,dx + \alpha \int_s^t \int_0^L xg(x,\sigma)\,dx\,d\sigma$$

From the above inequality, Definition 5.1 and the hypothesis (6.21) of g we may then let $L \to \infty$ and this will lead us to

$$\int_0^\infty xu(x,t)\,dx \leq \int_0^\infty xu(x,s)\,dx + \alpha \int_s^t \int_0^\infty xg(x,\sigma)\,dx\,d\sigma$$

and (6.23) is proved. Let us prove now that (6.26) holds, provided that the inequalities in (6.25) hold. Define another function

$$h_\varepsilon(x) := \min\{h(x), h(\varepsilon)\} \qquad \text{for } x \in (0,+\infty) \text{ and } \varepsilon \in (0,1) \tag{6.28}$$

Since h satisfies (6.24) it will result that h_ε is also a non-negative, non-increasing and bounded function on $(0,\infty)$ and it satisfies the subadditivity property. More precisely, the following hold $h_\varepsilon : (0,+\infty) \to \mathbb{R}_+$, $\forall \varepsilon \in (0,1)$; $h_\varepsilon(x) \leq h(\varepsilon)$ and as h is non-increasing we have

$$\begin{aligned} h_\varepsilon(x+y) &= \min\{h(x+y), h(\varepsilon)\} \leq \min\{h(x), h(\varepsilon)\} + \min\{h(y), h(\varepsilon)\} \\ &= h_\varepsilon(x) + h_\varepsilon(y) \Rightarrow \\ h_\varepsilon(x+y) &\leq h_\varepsilon(x) + h_\varepsilon(y), \forall (x,y) \in (0,\infty)^2 \end{aligned} \tag{6.29}$$

Moreover,

$$\lim_{\varepsilon \to 0} h_\varepsilon(x) = h(x) \tag{6.30}$$

Since $h_\varepsilon(x)$ satisfies (6.30) and $h_\varepsilon \in L^\infty(0,\infty)$ we may then choose $Q(x) = h_\varepsilon(x), \forall x \in (0,\infty)$ in (6.22). Using (6.29) we get

$$\begin{aligned} &\int_0^\infty h_\varepsilon(x)[u(x,t) - u(x,s)]\,dx \\ &= \frac{\alpha}{2} \int_s^t \int_0^\infty \int_0^\infty \phi(x,y)[h_\varepsilon(x+y) - h_\varepsilon(x) - h_\varepsilon(y)]u(x,\sigma)u(y,\sigma)\,dx\,dy\,d\sigma \\ &+ \alpha \int_s^t \int_0^\infty h_\varepsilon(x)g(x,\sigma)\,dx\,d\sigma \leq \alpha \int_s^t \int_0^\infty h_\varepsilon(x)g(x,\sigma)\,dx\,d\sigma \Rightarrow \\ &\int_0^\infty h_\varepsilon(x)u(x,t)\,dx \leq \int_0^\infty h_\varepsilon(x)u(x,s)\,dx + \alpha \int_s^t \int_0^\infty h_\varepsilon(x)g(x,\sigma)\,dx\,d\sigma \end{aligned} \tag{6.31}$$

Taking now $s \equiv 0$

$$\int_0^\infty h_\varepsilon(x)u(x,t)\,dx \le \int_0^\infty h_\varepsilon(x)u_0(x)\,dx + \alpha\int_0^t\int_0^\infty h_\varepsilon(x)g(x,\sigma)\,dx\,d\sigma$$

Now let $\varepsilon_n = \frac{1}{n} \to 0^+$ as $n \to \infty$ and since (6.30) holds, according to the Monotone Convergence Theorem (since $\{h_{\varepsilon_n}\}_{n\ge1}$ is a sequence of non-increasing functions, $h_{\varepsilon_n} \in L^\infty(0,\infty)$ and converges to h a.e.), then h is measurable and its integral is the limit of the integrals above, i.e. $\lim_{\varepsilon_n\to0^+}\int_0^\infty h_{\varepsilon_n}(x)u(x,t)\,dx = \int_0^\infty h(x)u(x,t)\,dx$. So $\lim_{\varepsilon_n\to0^+}\int_0^\infty h_{\varepsilon_n}(x)u_0(x)\,dx = \int_0^\infty h(x)u_0(x)\,dx$. Moreover, $\lim_{\varepsilon_n\to0^+}\int_0^\infty h_{\varepsilon_n}(x)g(x,\sigma)\,dx = \int_0^\infty h(x)g(x,\sigma)\,dx$ and (6.31) becomes

$$\int_0^\infty h(x)u(x,t)\,dx \le \int_0^\infty h(x)u_0(x)\,dx + \alpha\int_0^t\int_0^\infty h(x)g(x,\sigma)\,dx\,d\sigma \tag{6.32}$$

Since u_0 and g satisfy the integrability property (6.25) then $\int_0^\infty h(x)u(x,t)\,dx < \infty$ and u satisfies (6.26), (6.27). □

The following result illustrates the long-time behaviour of the total mass $M_1(t)$.

Theorem 6.3. Assume that $\phi(x,y)$ satisfies (6.19). Let $u_0 \in X^+$, g satisfies (6.21) and u be a solution to the equation (6.16) on $[0,\infty)$ with initial condition u_0. Then

$$\begin{aligned} M_1(t) \le{}& \frac{\sqrt{2\alpha}}{\sqrt{t}}\frac{1}{\beta}\Big(\int_0^\infty u_0(x)\,dx + \alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\Big)^{\frac12} \\ &+ \frac{\alpha}{t}\int_0^t\int_s^t\int_0^\infty xg(x,\tau)\,dx\,d\tau\,ds \end{aligned} \tag{6.33}$$

and

$$T_{gel} \le \frac{2\alpha}{\beta^2}\cdot\frac{\Big\{\int_0^\infty u_0(x)\,dx + \alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\Big\}}{\Big\{M_1(0) + \alpha\int_0^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma\Big\}^2} \tag{6.34}$$

Proof. Let $s \in (0,\infty)$ and $t \in (s,\infty)$. Take $Q \equiv 1$ in (6.22) then, since $\theta(x) = 1 + \frac{\beta}{\alpha}x > \frac{\beta}{\alpha}x$ then (6.22) becomes

$$\begin{aligned} \int_0^\infty [u(x,t) - u(x,s)]\,dx = {}& -\frac{\alpha}{2}\int_s^t\int_0^\infty\int_0^\infty \phi(x,y)u(x,\sigma)u(y,\sigma)\,dx\,dy\,d\sigma \\ &+ \alpha\int_s^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma \end{aligned} \tag{6.35}$$

Since $\phi(x,y) = \theta(x)\theta(y)$ then

$$\Big(\int_0^\infty \theta(x)u(x,\sigma)\,dx\Big)^2 > \Big(\int_0^\infty \frac{\beta}{\alpha}xu(x,\sigma)\,dx\Big)^2$$
$$= \frac{\beta^2}{\alpha^2}\Big(\int_0^\infty xu(x,\sigma)\,dx\Big)^2 = \frac{\beta^2}{\alpha^2}M_1^2(\sigma)$$

Therefore,

$$-\frac{\alpha}{2}\int_s^t\int_0^\infty\int_0^\infty \phi(x,y)u(x,\sigma)u(y,\sigma)\,dx\,dy\,d\sigma = -\frac{\alpha}{2}\int_s^t\Big(\int_0^\infty \theta(x)u(x,\sigma)\,dx\Big)^2 d\sigma$$
$$< -\frac{\alpha}{2}\frac{\beta^2}{\alpha^2}\int_s^t M_1^2(\sigma)\,d\sigma$$

Thus (6.35) becomes

$$\int_0^\infty [u(x,t)-u(x,s)]\,dx < -\frac{1}{2}\frac{\beta^2}{\alpha}\int_s^t M_1^2(\sigma)\,d\sigma + \int_s^t\int_0^\infty \alpha g(x,\sigma)\,dx\,d\sigma \qquad \text{or}$$
$$\frac{1}{2}\Big(\frac{\beta^2}{\alpha}\Big)\int_s^t M_1^2(\sigma)\,d\sigma < -\int_0^\infty [u(x,t)-u(x,s)]\,dx + \alpha\int_s^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma$$
$$= -\int_0^\infty u(x,t)\,dx + \int_0^\infty u(x,s)\,dx + \alpha\int_s^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma$$
$$\le \int_0^\infty u(x,s)\,dx + \alpha\int_s^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma$$

where we removed the negative terms.

Multiplying the above inequality by $2\big(\frac{\alpha}{\beta^2}\big)$ we get

$$\int_s^t M_1^2(\sigma)\,d\sigma \le 2\Big(\frac{\alpha}{\beta^2}\Big)\Big[\int_0^\infty u(x,s)\,dx + \alpha\int_s^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\Big] \qquad \text{take } s\equiv 0 \tag{6.36}$$

$$\int_0^t M_1^2(\sigma)\,d\sigma \le 2\Big(\frac{\alpha}{\beta^2}\Big)\int_0^\infty u_0(x)\,dx + 2\Big(\frac{\alpha}{\beta}\Big)^2\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma \tag{6.37}$$

Since $\sigma \le t$ then from (6.23) we have

$$M_1(t) \le M_1(\sigma) + \alpha\int_\sigma^t\int_0^\infty xg(x,\tau)\,dx\,d\tau \Rightarrow \qquad \text{integrating it w.r.t.} \sigma\in[0,t] \Rightarrow$$
$$tM_1(t) \le \int_0^t M_1(s)\,ds + \alpha\int_0^t\int_\sigma^t\int_0^\infty xg(x,\tau)\,dx\,d\tau\,d\sigma \tag{6.38}$$

Denote by

$$\rho(t) := \int_0^t M_1(s)\,ds \qquad \text{for } t\in\mathbf{R}_+ \tag{6.39}$$

Using Hölder's inequality we obtain:

$$\int_0^t M_1(s)\,ds \le \Big(\int_0^t 1^2\,ds\Big)^{\frac{1}{2}} \cdot \Big(\int_0^t M_1^2(s)\,ds\Big)^{\frac{1}{2}} = \sqrt{t}\Big(\int_0^t [M_1(s)]^2\,ds\Big)^{\frac{1}{2}}$$

By using (6.37) for the right-hand side term above we obtain

$$\begin{aligned}\int_0^t M_1(s)\,ds &\le \sqrt{t}\cdot\left\{2\frac{\alpha}{\beta^2}\int_0^\infty u_0(x)\,dx + 2\Big(\frac{\alpha}{\beta}\Big)^2\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\right\}^{\frac{1}{2}}\\ &= \frac{\sqrt{2\alpha t}}{\beta}\left\{\int_0^\infty u_0(x)\,dx + \alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\right\}^{\frac{1}{2}}\end{aligned}$$

Using the inequality (6.38) we have

$$\begin{aligned}tM_1(t) &\le \int_0^t M_1(s)\,ds + \alpha\int_0^t\int_s^t\int_0^\infty xg(x,\tau)\,dx\,d\tau\,ds\\ &\le \frac{\sqrt{2\alpha t}}{\beta}\left\{\int_0^\infty u_0(x)\,dx + \alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\right\}^{\frac{1}{2}}\\ &\quad + \alpha\int_0^t\int_s^t\int_0^\infty xg(x,\tau)\,dx\,d\tau\,ds\end{aligned}$$

Dividing this last inequality by $t > 0$, we get inequality (6.33).

Theorem 6.3 also gives an upper bound on the gelation time T_{gel} which can be defined by

$$T_{gel} := \inf\left\{t \in \mathbb{R}_+ : M_1(t) < M_1(0) + \alpha\int_0^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma\right\} \in [0,+\infty]$$

The gelation time T_{gel} is not greater than the root t of the following inequality:

$$\begin{aligned}&\frac{\sqrt{2\alpha}}{\sqrt{t}}\frac{1}{\beta}\left\{\int_0^\infty u_0(x)\,dx + \alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\right\}^{\frac{1}{2}}\\ &\quad + \frac{\alpha}{t}\int_0^t\int_s^t\int_0^\infty xg(x,\tau)\,dx\,d\tau\,ds < M_1(0) + \alpha\int_0^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma \Leftrightarrow\\ &\frac{\sqrt{2\alpha}}{\sqrt{t}}\frac{1}{\beta}\left\{\int_0^\infty u_0(x)\,dx + \alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\right\}^{\frac{1}{2}}\\ &\quad < \Big[M_1(0) + \alpha\int_0^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma\Big] - \frac{\alpha}{t}\int_0^t\int_s^t\int_0^\infty xg(x,\tau)\,dx\,d\tau\,ds\end{aligned}$$

By removing the last term $\Big(-\frac{\alpha}{t}\int_0^t\int_s^t\int_0^\infty xg(x,\tau)\,dx\,d\tau\,ds\Big)$ as it is non-positive we obtain

$$\frac{\sqrt{2\alpha}}{\sqrt{t}}\frac{1}{\beta}\Big\{\int_0^\infty u_0(x)\,dx+\alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\Big\}^{\frac{1}{2}}$$
$$<\Big[M_1(0)+\alpha\int_0^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma\Big]\Leftrightarrow$$

$$\frac{2\alpha}{t}\frac{1}{\beta^2}\Big\{\int_0^\infty u_0(x)\,dx+\alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\Big\}$$
$$<\Big[M_1(0)+\alpha\int_0^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma\Big]^2\Leftrightarrow$$

$$t\geq\frac{2\alpha}{\beta^2}\cdot\frac{\Big(\int_0^\infty u_0(x)\,dx+\alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\Big)}{\Big(M_1(0)+\alpha\int_0^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma\Big)^2}$$

From the definition of T_{gel} it will result that an upper bound for T_{gel} would be

$$T_{gel}\leq 2\Big(\frac{\alpha}{\beta^2}\Big)\frac{\Big(\int_0^\infty u_0(x)\,dx+\alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\Big)}{\Big(M_1(0)+\alpha\int_0^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma\Big)^2}$$

In the following our goal is to extend the results (B), (C) of the Theorem 6.1 in Section 6.1 in order to analyse the temporal decay of the total mass of the solutions to the equation (6.16).

Theorem 6.4. If $u_0\in X^+$ and g satisfy, in addition to (6.21) and the assumptions in Theorem 6.3, the integrability conditions

$$I_q:=\int_0^\infty x^{-q}u_0(x)\,dx<\infty\ \text{ and }\ G_q(t):=\int_0^\infty x^{-q}g(x,t)\,dx<\infty\ \text{ for some } q>0$$

and for any $t\in\mathbb{R}_+$ then

$$M_1(t)\leq\frac{\sqrt{2\alpha}}{\beta\sqrt{t}}\Big\{M_1(0)^{\frac{q}{q+1}}\Big(I_q+\alpha\int_0^t G_q(\sigma)\,d\sigma)\Big)^{\frac{1}{q+1}}$$
$$+\Big(\frac{\alpha}{\beta}\Big)^m\int_0^t\Big[G_q(\sigma)\Big]^{\frac{1}{q+1}}\|g(\sigma)\|_X^{\frac{q}{q+1}}\,d\sigma\Big\}^{\frac{1}{2}}$$
$$+\frac{\alpha^2}{\beta t}\int_0^t\int_s^t\|g(\sigma)\|_X\,d\sigma\,ds \tag{6.40}$$

where m is defined as $m=\frac{2q+1}{q+1}$.

Proof The function $h : (0,\infty) \to [0,\infty)$ defined by $h(x) = x^{-q}$ satisfies the assumptions in Lemma 6.4 above which means, h is a non-negative, non-increasing function so that $(x+y)^{-q} \le x^{-q} + y^{-q}, \forall (x,y) \in (0,\infty)^2$. Therefore, according to Lemma 6.4, (6.27) holds for $h(x) = x^{-q}$, i.e.

$$\int_0^\infty x^{-q} u(x,t)\,dx \le \int_0^\infty x^{-q} u_0(x)\,dx + \alpha \int_0^t \int_0^\infty x^{-q} g(x,\sigma)\,dx\,d\sigma$$
$$= I_q + \alpha \int_0^t G_q(\sigma)\,d\sigma \tag{6.41}$$

We can write: $\int_0^\infty u(x,s)\,dx = \int_0^\infty \big(xu(x,s)\big)^{\frac{q}{q+1}} \big(x^{-q}u(x,s)\big)^{\frac{1}{q+1}}\,dx$. By Hölder's inequality: $\int_0^\infty f(x)g(x)\,dx \le \Big(\int_0^\infty [f(x)]^a\,dx\Big)^{\frac{1}{a}} \cdot \Big(\int_0^\infty [g(x)]^b\,dx\Big)^{\frac{1}{b}}$ with $f(x) = \big(xu(x,s)\big)^{\frac{q}{q+1}}$, $g(x) = \big(x^{-q}u(x,s)\big)^{\frac{1}{q+1}}$ and $a = \frac{q+1}{q}$, $b = q+1$ are strictly positive numbers satisfying $\frac{1}{a} + \frac{1}{b} = 1$, it will then follow

$$\int_0^\infty u(x,s)\,dx \le \Big(\int_0^\infty xu(x,s)\,dx\Big)^{\frac{q}{q+1}} \cdot \Big(\int_0^\infty x^{-q}u(x,s)\,dx\Big)^{\frac{1}{q+1}}$$
$$= \big[M_1(s)\big]^{\frac{q}{q+1}} \cdot \Big(\int_0^\infty x^{-q}u(x,s)\,dx\Big)^{\frac{1}{q+1}}$$
$$\le \big[M_1(s)\big]^{\frac{q}{q+1}} \cdot \Big(I_q + \alpha \int_0^t G_q(\sigma)\,d\sigma\Big)^{\frac{1}{q+1}} \tag{6.42}$$

If we let $s \equiv 0$ in (6.42) then the first integral in (6.37) becomes:

$$\int_0^\infty u_0(x)\,dx \le \big[M_1(0)\big]^{\frac{q}{q+1}} \cdot \Big(I_q + \alpha \int_0^t G_q(\sigma)\,d\sigma\Big)^{\frac{1}{q+1}} \tag{6.43}$$

Using Hölder's inequality again for the integral $\int_0^\infty g(x,\sigma)\,dx$ in the right-hand side of (6.37) and the definition of G_q we obtain

$$\int_0^\infty g(x,\sigma)\,dx \le \Big(\int_0^\infty xg(x,\sigma)\,dx\Big)^{\frac{q}{q+1}} \cdot \Big[G_q(\sigma)\Big]^{\frac{1}{q+1}} \tag{6.44}$$

Substituting inequalities (6.43) and (6.44) into (6.37) we obtain

$$\int_0^t M_1^2(\sigma)\,d\sigma \le 2\frac{\alpha}{\beta^2}\Bigg\{ M_1(0)^{\frac{q}{q+1}} \Big(I_q + \alpha \int_0^t G_q(\sigma)\,d\sigma\Big)^{\frac{1}{q+1}}$$
$$+ \alpha \int_0^t \Big[G_q(\sigma)\Big]^{\frac{1}{q+1}} \Big(\int_0^\infty xg(x,\sigma)\,dx\Big)^{\frac{q}{q+1}}\,d\sigma \Bigg\}$$

Moreover, since $g \in L^1(0,\infty;X) \Rightarrow \int_0^\infty xg(x,\sigma)\,dx < \frac{\alpha}{\beta}\int_0^\infty (1+\frac{\beta}{\alpha}x)g(x,\sigma)\,dx = \frac{\alpha}{\beta}\|g(\sigma)\|_X < \infty : (*)$ then we have

$$\int_0^t M_1^2(\sigma)\,d\sigma \le 2\frac{\alpha}{\beta^2}\Big\{ M_1(0)^{\frac{q}{q+1}}\Big(I_q + \alpha\int_0^t G_q(\sigma)\,d\sigma\Big)^{\frac{1}{q+1}} + \Big(\frac{\alpha}{\beta}\Big)^m \int_0^t \Big[G_q(\sigma)\Big]^{\frac{1}{q+1}} \|g(\sigma)\|_X^{\frac{q}{q+1}}\,d\sigma\Big\} \tag{6.45}$$

where m was defined above. Let $t \in (0,\infty)$. On the one hand, using Hölder's inequality we get $[\rho(t)]^2 \le t\int_0^t [M_1(s)]^2\,ds$ where ρ was defined in (6.39). On the other hand, multiplying (6.45) by t and extracting the radical we get

$$\rho(t) \le \frac{\sqrt{2\alpha t}}{\beta}\Big\{ M_1(0)^{\frac{q}{q+1}}\Big(I_q + \alpha\int_0^t G_q(\sigma)\,d\sigma\Big)^{\frac{1}{q+1}} + \Big(\frac{\alpha}{\beta}\Big)^m \int_0^t \Big[G_q(\sigma)\Big]^{\frac{1}{q+1}} \|g(\sigma)\|_X^{\frac{q}{q+1}}\,d\sigma\Big\}^{\frac{1}{2}}$$

Since by Lemma 6.4, (6.23) holds, then integrating it over $[0,t]$ we get

$$tM_1(t) \le \int_0^t M_1(s)\,ds + \alpha\int_0^t\int_s^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma\,ds$$

Or dividing it by t we obtain

$$M_1(t) \le \frac{\sqrt{2\alpha}}{\beta\sqrt{t}}\Big\{ M_1(0)^{\frac{q}{q+1}}\Big(I_q + \alpha\int_0^t G_q(\sigma)\,d\sigma\Big)^{\frac{1}{q+1}} + \Big(\frac{\alpha}{\beta}\Big)^m \int_0^t \Big[G_q(\sigma)\Big]^{\frac{1}{q+1}} \|g(\sigma)\|_X^{\frac{q}{q+1}}\,d\sigma\Big\}^{\frac{1}{2}} + \frac{\alpha}{t}\int_0^t\int_s^t\int_0^\infty xg(x,\sigma)\,dx\,d\sigma\,ds \tag{6.46}$$

and using relation (*) above we obtain that the conclusion (6.40) of the theorem holds for any $t \in (0,\infty)$. □

Theorem 6.5. For any solution to the equation (6.16) with initial data $u_0 \in X^+$ such that $u_0(x) = 0$, for $x \in [0,\delta]$ and some $\delta > 0$ and for which the source term satisfies the hypothesis (6.21) and $g(x,\sigma) = 0$, for any $x \in [0,\delta]$, $\forall\sigma \in \mathbb{R}_+$ we have

$$M_1(t) \le \frac{\sqrt{2\alpha}}{\beta\sqrt{\delta t}}\Big(M_1(0) + \frac{\alpha^2}{\beta}\int_0^t \|g(\sigma)\|_X\,d\sigma\Big)^{\frac{1}{2}} + \frac{\alpha^2}{\beta t}\int_0^t\int_s^t \|g(\sigma)\|_X\,d\sigma\,ds \tag{6.47}$$

Proof We have already proved that the following inequality in Lemma 6.1 holds: For any $Q \in L^\infty(0,\infty)$

$$\int_0^\infty Q(x)[u(x,t)-u_0(x)]\,dx = \frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty \widetilde{Q}(x,y)\phi(x,y)u(x,\sigma)u(y,\sigma)\,dx\,dy\,d\sigma + \alpha\int_0^t\int_0^\infty Q(x)g(x,\sigma)\,dx\,d\sigma \tag{6.48}$$

where $\widetilde{Q}(x,y) = \alpha(Q(x+y)-Q(x)-Q(y)), \forall x,y \geq 0$. Take $Q(x) = \chi_{[0,\delta]}(x)$ then $\widetilde{Q}(x,y) \leq 0$, for any $x,y \in [0,\infty)$ and equality (6.48) becomes

$$\int_0^\delta u(x,t)\,dx - \int_0^\delta u_0(x)\,dx = -\frac{\alpha}{2}\int_0^t\int_0^\delta\int_0^\delta \phi(x,y)u(x,\sigma)u(y,\sigma)\,dx\,dy\,d\sigma + \alpha\int_0^t\int_0^\delta g(x,\sigma)\,dx\,d\sigma \leq \alpha\int_0^t\int_0^\delta g(x,\sigma)\,dx\,d\sigma$$

Since $u_0 \equiv 0$ on $[0,\delta]$ and $g(\cdot,\sigma) \equiv 0$ on $[0,\delta]$, for any $\sigma \in \mathbb{R}_+$ then

$$0 \leq \int_0^\delta u(x,t)\,dx \leq \int_0^\delta u_0(x)\,dx + \int_0^t\int_0^\delta \alpha g(x,\sigma)\,dx\,d\sigma = 0$$

we obtain $\int_0^\delta u(x,t)\,dx = 0$, for every time $t \in \mathbb{R}_+$. Hence $u(x,t) = 0$, a.e. in $[0,\delta]$. Therefore,

$$M_0(t) = \int_0^\infty u(x,t)\,dx = \int_\delta^\infty u(x,t)\,dx = \int_\delta^\infty \frac{x}{x}\cdot u(x,t)\,dx \leq \frac{1}{\delta}\int_\delta^\infty xu(x,t)\,dx \leq \frac{1}{\delta}M_1(t)$$

Therefore,

$$M_0(t) \leq \frac{1}{\delta}M_1(t) \qquad \forall t \in \mathbb{R}_+ \tag{6.49}$$

Let $t \in (0,+\infty)$. On the one hand, it follows from Hölder inequality that

$$[\rho(t)]^2 \leq t\int_0^t [M_1(s)]^2\,ds$$

where ρ was defined in (6.39). On the other hand, we have proved already that (6.36) holds and together with (6.49) and since $\int_0^\infty g(x,\sigma)\,dx = \int_\delta^\infty \frac{x}{x}g(x,\sigma)\,d\sigma \leq \frac{1}{\delta}\int_\delta^\infty xg(x,\sigma)\,dx$ then

$$\int_s^t M_1^2(\sigma)\,d\sigma \leq 2\frac{\alpha}{\beta^2}\Big(\frac{1}{\delta}M_1(s) + \alpha\int_s^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\Big) \qquad \text{(take now } s=0\text{)}$$

$$\int_0^t M_1^2(\sigma)\,d\sigma \leq 2\frac{\alpha}{\beta^2}\Big(\frac{1}{\delta}M_1(0) + \alpha\int_0^t\int_0^\infty g(x,\sigma)\,dx\,d\sigma\Big)$$

$$\rho(t) \leq \frac{\sqrt{2\alpha t}}{\sqrt{\beta^2\delta}}\Big(M_1(0) + \alpha\int_0^t\int_\delta^\infty xg(x,\sigma)\,dx\,d\sigma\Big)^{\frac{1}{2}}$$

Since $M_1(t) \leq M_1(s) + \alpha \int_s^t \int_0^\infty x g(x,\sigma)\, dx\, d\sigma$ then integrating it with respect to $\sigma \in [0,t]$ we get

$$tM_1(t) \leq \rho(t) + \alpha \int_0^t \int_s^t \int_0^\infty x g(x,\sigma)\, dx\, d\sigma\, ds \tag{6.50}$$

Dividing it by t we get

$$\begin{aligned} M_1(t) &\leq \frac{\sqrt{2\alpha}}{\sqrt{\delta t}} \frac{1}{\beta} \Big(M_1(0) + \alpha \int_0^t \int_\delta^\infty x g(x,\sigma)\, dx\, d\sigma \Big)^{\frac{1}{2}} + \frac{\alpha}{t} \int_0^t \int_s^t \int_\delta^\infty x g(x,\tau)\, dx\, d\tau\, ds \\ &\leq \frac{\sqrt{2\alpha}}{\sqrt{\delta t}} \frac{1}{\beta} \Big(M_1(0) + \frac{\alpha^2}{\beta} \int_0^t \int_0^\infty (1 + \frac{\beta}{\alpha} x) g(x,\sigma)\, dx\, d\sigma \Big)^{\frac{1}{2}} \\ &+ \frac{\alpha^2}{\beta t} \int_0^t \int_s^t \int_0^\infty (1 + \frac{\beta}{\alpha} x) g(x,\tau)\, dx\, d\tau\, ds \\ &= \frac{\sqrt{2\alpha}}{\beta\sqrt{\delta t}} \Big(M_1(0) + \frac{\alpha^2}{\beta} \int_0^t \|g(\sigma)\|_X\, d\sigma \Big)^{\frac{1}{2}} + \frac{\alpha^2}{\beta t} \int_0^t \int_s^t \|g(\sigma)\|_X\, d\sigma\, ds \end{aligned} \tag{6.51}$$

Therefore the proof of the theorem is complete. □

It follows from Theorem 6.5 that the temporal decay of the total total mass depends strongly on the amount of clusters of very small size ($x \approx 0$) in the initial distribution and in the source term g and the smaller these quantities are, the faster is the decay of the total mass M_1.

6.3 A lower bound on the gelation time.

The main purpose of this section is to obtain a lower bound on the sourceless gelation time using the method employed by Shirvani and van Roessel in [39] for the pure coagulation equation.

Before stating the main theorem of this section (Theorem 6.8), we present some results obtained in [39] including some steps in obtaining the gelation time for the Smoluchowski coagulation equation:

$$\frac{\partial c}{\partial t}(x,t) = \frac{1}{2} \int_0^x K(x-y,y) c(x-y,t) c(y,t)\, dy - \int_0^\infty K(x,y) c(x,t) c(y,t)\, dy \tag{6.52}$$

subject to the initial condition

$$c(x,0) = c_0(x) \geq 0 \tag{6.53}$$

The coagulation kernel of interest in this chapter as well as in [39] satisfies the hypotheses $K \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, (4.3), (4.4) and contains both the constant and product kernels as particular cases:

$$K(x,y) = \theta(x)\theta(y), \text{ where } \theta(x) = \alpha + \beta x,\ \alpha, \beta \geq 0 \tag{6.54}$$

One of the main objectives in [39] was to determine necessary and sufficient conditions under which the solution of (6.52) is mass-conserving and also to determine an explicit formula for the total mass for all times. In order to accomplish this, the method of Laplace transforms was used. Below we present a formal derivation of the partial differential equation obtained by using the method in [39].

Formal derivation of the pde.
If we let

$$y(x,t) := \theta(x)c(x,t) \text{ and} \tag{6.55}$$

$$N(t) := \int_0^\infty y(x,t)\,dx \tag{6.56}$$

then multiplying equation (6.52) by $\theta(x)$, (6.52) may be written as

$$\frac{\partial y}{\partial t}(x,t) = \frac{\theta(x)}{2}(y * y)(x,t) - N(t)\theta(x)y(x,t) \tag{6.57}$$

where " $*$ " denotes the convolution product defined by

$$(c * c)(x,t) := \int_0^x c(x-y,t)c(y,t)\,dy$$

Throughout this section we assume that N satisfies the following hypothesis $N \in L^1_+(0,R)$,
$\forall R > 0$, where $L^1(0,R)$ was defined in (2.9), Chapter 2.

Now, if we let $Y(z,t) := \mathcal{L}[y(x,t)](z,t) = \int_0^\infty e^{-zx}y(x,t)\,dx$ then, by formally taking the Laplace transform, equation (6.57) takes the form:

$$\frac{\partial Y}{\partial t}(z,t) = \frac{1}{2}\mathcal{L}\{\theta(x)(y * y)(x,t)\}(z,t) - N(t)\mathcal{L}\{\theta(x)y(x,t)\}(z,t) \tag{6.58}$$

In the case $\theta(x) = \alpha + \beta x$, with $\alpha, \beta \geq 0$ constants, using the following properties on the Laplace transform,

$$\mathcal{L}[u * v] = \mathcal{L}(u) \cdot \mathcal{L}(v)$$

$$\mathcal{L}\{xu(x,t)\}(z,t) = -\frac{\partial \mathcal{L}\{u(x,t)\}}{\partial z}(z,t)$$

$$\mathcal{L}[\theta(x)r(x)](z) = \alpha\mathcal{L}[r(x)](z) - \beta\frac{\partial}{\partial z}\mathcal{L}[r(x)](z), \text{for any function } r(x)$$

we get

$$\mathcal{L}[\theta(x)y(x,t)](z,t) = \alpha Y(z,t) - \beta\frac{\partial Y}{\partial z}(z,t)$$

$$\mathcal{L}\{\theta(x)(y * y)(x,t)\}(z,t) = \alpha Y^2(z,t) - \beta\frac{\partial}{\partial z}Y^2(z,t) = \alpha Y^2(z,t) - 2\beta Y(z,t)\frac{\partial Y}{\partial z}(z,t) \tag{6.59}$$

equation (6.58) becomes

$$\beta\big(Y(z,t)-N(t)\big)\frac{\partial Y}{\partial z}(z,t)+\frac{\partial Y}{\partial t}(z,t)=\frac{\alpha}{2}Y^2(z,t)-\alpha N(t)Y(z,t) \tag{6.60}$$

subject to the initial condition

$$Y(z,0)=\mathcal{L}[y_0(x)](z,0)=\mathcal{L}[y_0](z) \tag{6.61}$$

where $y_0(x)=y(x,0)$. Consider the first-order partial differential equation

$$\begin{cases} \beta(u-N(t))\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}=\frac{\alpha}{2}u^2-\alpha N(t)u \\ u(z,0)=h(z)=\mathcal{L}[y_0(x)](z,0) \end{cases} \tag{6.62}$$

The characteristic equations associated with (6.62) are:

$$\frac{dz}{ds}=\beta(w-N(t)), \qquad z|_{s=0}=\xi>0, \tag{6.63}$$

$$\frac{dt}{ds}=1, \qquad t|_{s=0}=0, \tag{6.64}$$

$$\frac{dw}{ds}=\frac{\alpha}{2}w^2-\alpha N(t)w, \qquad w|_{s=0}=h(\xi)\geq 0. \tag{6.65}$$

Clearly, $t=s$. Two separate cases may be considered here ($\alpha\neq 0$ and $\alpha=0$). Let us assume that $\alpha\neq 0$ holds. Let r be the function of time defined by

$$\frac{r''(t)}{r'(t)}=-\alpha N(t)$$

$$r(0)=0;\ r'(0)=\frac{\alpha}{2}$$

Then $r(t)=\frac{\alpha}{2}\int_0^t e^{-\alpha\int_0^\lambda N(\mu)\,d\mu}\,d\lambda$ is a well-defined function and invertible for all $t\geq 0$, since $N\in L^1_+(0,R)$, for every $R>0$ and $r'(t)>0$. Since equation (6.65) is a Bernoulli equation, then is readily solved. The solution is $w=W(\xi,t)=\frac{2}{\alpha}\frac{h(\xi)r'(t)}{1-h(\xi)r(t)}$. Moreover, it was proved in [39] that Z is a completely monotonic function. Then equation (6.63) has the following solution:

$$z=Z(\xi,t)=\xi-\frac{2\beta}{\alpha}\ln(1-h(\xi)r(t))+\frac{\beta}{\alpha}\ln\Big(\frac{2r'(t)}{\alpha}\Big) \tag{6.66}$$

The circumstances under which (6.66) can be inverted for all ξ were investigated in order to get ξ as a function of z and t.

Remark. For the pure coagulation equation (6.52) it was proved that, if the boundary condition h of the pde (6.62) is completely monotonic then the solution to the pde is completely monotonic for all time. The time (if any) at which the gel forms is obtained at this stage.
Moreover, the following theorem was proved in [39]:

Theorem 6.6. If (i) $N \in L^1_+(0,R)$ for all $R > 0$; (ii) α and β are non-negative constants;
(iii) h is completely monotonic for $z \geq 0$ and satisfies $h(0) = N_0 := N(0)$, then equation (6.62) with initial condition $u(z,0) = h(z)$ has a solution u valid for $(z,t) \in (0,\infty) \times [0,T_0)$ which is completely monotonic in $z \in (0,\infty)$ for each $t \in [0,T_0)$, where

(a) $T_0 = +\infty$ if $\beta = 0$ (no gel);

(b) $T_0 = 0$ if $\beta \neq 0$ and $h'(0^+) = -\infty$ (instantaneous gel);

(c) $0 < T_0 < \infty$ if $\beta \neq 0$ and $|h'(0^+)| < \infty$.

The gelation time in the pure coagulation equation (6.52), we call the "old gel time". This is denoted by T_0, and was proved to be the first instance when the system of equations (6.66) can no longer be inverted. More precisely,
$T_0 := \inf\left\{t > 0 : \text{such that it satisfies the equation } D(\xi,t) := \frac{\partial Z}{\partial \xi}(\xi,t) = 0\right\}$.

Since in the pure coagulation equation (6.52), the value of $D(\xi,t)$ can be determined exactly then

$$t = \frac{-2}{\alpha\big(h(0) - h(\xi)\big) + 2\beta h'(\xi)} := T(\xi)$$

By setting $N(t) = u(0,t)$ it was obtained that $r(t) = \frac{\alpha t}{2+\alpha N_0 t} < \frac{1}{N_0}$. According to the definition of the old gelation time, we have

$$T_0 = \min_{\xi>0} T(\xi) = \min_{\xi>0} \frac{-2}{\alpha\big(h(0) - h(\xi)\big) + 2\beta h'(\xi)}$$

Since h is completely monotonic (in the sense of Definition 2.12) and α, β are positive constants with $\beta \neq 0$ then it follows

$$T'(\xi) = \frac{2\big[2\beta h''(\xi) - \alpha h'(\xi)\big]}{\big[\alpha\big(h(0) - h(\xi)\big) + 2\beta h'(\xi)} > 0$$

which means $T(\xi)$ is an increasing function of ξ and thus for any $\xi > 0 \Rightarrow T(\xi) \geq T(0) = T_0 = \frac{-1}{\beta h'(0^+)} = T_{gel}$. This means that the number of zeros in the pure coagulation equation (6.52) which we'll denote by N^* is $N^* = 1$. In [39] it was proved that before the formation of the gel time T_0, the total mass $M_1(t)$ is conserved. Moreover, the main theorem in [39] can be stated here:

Theorem 6.7. Let $C_0(z) = \mathcal{L}[c_0](x)$. If $\theta(x)c_0(x) \in L^1(0,\infty)$ and $c_0(x) \geq 0$, for $x \in [0,\infty)$ then the coagulation equation (6.52) has a mass conserving solution for all $t \in [0,T_0)$ where:

(a) $T_0 = +\infty$ (no gel) if $\beta = 0$

(b) $T_0 = 0$ (instantaneous gelation) if $\beta \neq 0$ and $C_0''(0^+) = +\infty$

(c) $T_0 = 1/(\beta(\alpha + \beta C_0''(0^+)) < \infty$ if $\beta \neq 0$ and $C_0''(0^+) < \infty$ where $C_0''(0^+) = \int_0^\infty x^2 c_0(x)\,dx$

Next, we are interested in the coagulation equation with source terms. Our main purpose is to obtain a lower bound on the sourceless gelation time $T_{gel} := \inf\{t > 0 : M_1(t) < M_1(0)\}$ in the pure coagulation equation.

Let us first consider the following coagulation equation with a source term

$$\frac{\partial c}{\partial t}(x,t) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)\,dy - \int_0^\infty K(x,y)c(x,t)c(y,t)\,dy + g(x,t) \tag{6.67}$$

subject to the same initial condition

$$c(x,0) = c_0(x) \geq 0 \tag{6.68}$$

where the coagulation kernel K satisfies the hypotheses (6.54). Using the same notations (6.55) and (6.56) as in the pure coagulation equation then equation (6.67) may be written as

$$\frac{\partial y}{\partial t}(x,t) = \frac{\theta(x)}{2}(y * y)(x,t) - N(t)\theta(x)y(x,t) + g(x,t)\theta(x) \tag{6.69}$$

Now, if we let $Y(z,t) = \mathcal{L}[y(x,t)](z,t) := \int_0^\infty e^{-zx}y(x,t)\,dx$ then by formally taking the Laplace transform, equation (6.69) may be written as

$$\frac{\partial Y}{\partial t}(z,t) = \frac{1}{2}\mathcal{L}[\theta(x)(y*y)(x,t)](z,t) - N(t)\mathcal{L}[\theta(x)y(x,t)](z,t) + \mathcal{L}[g(x,t)\theta(x)](z,t) \tag{6.70}$$

Using properties of Laplace transform mentioned above, and denoting by

$$F(z,t) := \mathcal{L}[g(x,t)\theta(x)](z,t) = \int_0^\infty e^{-zx}g(x,t)\theta(x)\,dx$$

we get

$$\mathcal{L}[\theta(x)g(x,t)](z,t) = \alpha\mathcal{L}[g(x,t)](z,t) - \beta\frac{\partial}{\partial z}\mathcal{L}[g(x,t)](z,t)$$

Equation (6.70) becomes

$$\beta\Big(Y(z,t) - N(t)\Big)\frac{\partial Y}{\partial z}(z,t) + \frac{\partial Y}{\partial t}(z,t) = \frac{\alpha}{2}Y^2(z,t) - \alpha N(t)Y(z,t) + F(z,t) \tag{6.71}$$

subject to the initial condition

$$Y(z,0) = \mathcal{L}[y_0(x)](z,0) = \mathcal{L}[y_0](z) \tag{6.72}$$

where $y_0(x) = y(x,0)$. Consider the first-order partial differential equation

$$\begin{cases} \beta(u - N(t))\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = \frac{\alpha}{2}u^2 - \alpha N(t)u + F(z,t) \\ u(z,0) = h(z) = \mathcal{L}[y_0](z) \end{cases} \tag{6.73}$$

The characteristic equations associated with (6.73) are

$$\frac{dz}{ds} = \beta(w - N(t)), \qquad z|_{s=0} = \xi > 0, \tag{6.74}$$

$$\frac{dt}{ds} = 1, \qquad t|_{s=0} = 0, \tag{6.75}$$

$$\frac{dw}{ds} = \frac{\alpha}{2}w^2 - \alpha N(t)w - F(z,t), \qquad w|_{s=0} = h(\xi) \geq 0. \tag{6.76}$$

Clearly, $t = s$. Since both z and w depend on ξ and t the system above can be rewritten as

$$\frac{\partial Z}{\partial t}(\xi,t) = \beta(W(\xi,t) - N(t)), \qquad Z(\xi,0) = \xi, \tag{6.77}$$

$$\frac{\partial W}{\partial t}(\xi,t) = \frac{\alpha}{2}W^2(\xi,t) - \alpha N(t)W(\xi,t) + F(Z(\xi,t),t), \qquad W(\xi,0) = h(\xi). \tag{6.78}$$

In particular, it is possible to set

$$N(t) = u(0,t) = W(0,t)$$

and therefore, the above system of equations takes the form:

$$\frac{dW}{dt}(0,t) = \frac{\alpha}{2}W^2(0,t) - \alpha N(t)W(0,t) + F(Z(0,t),t) \Rightarrow$$

$$\frac{dN(t)}{dt} = \frac{\alpha}{2}N^2(t) - \alpha N^2(t) + F(0,t), \text{with } N(0) = h(0) = N_0$$

Let us denote by $Q(t) := F(0,t)$ then we get

$$\begin{aligned} \frac{dN(t)}{dt} &= -\frac{\alpha}{2}N^2(t) + Q(t) \\ N(0) &= h(0) := N_0 \end{aligned} \tag{6.79}$$

We also denote by

$$\begin{cases} D(\xi,t) & := \frac{\partial Z(\xi,t)}{\partial \xi} \\ E(\xi,t) & := \frac{\partial W(\xi,t)}{\partial \xi} \end{cases}$$

Differentiating now (6.77) with respect to ξ we get

$$\frac{\partial^2 Z}{\partial \xi \partial t}(\xi,t) = \beta \frac{\partial W}{\partial \xi}(\xi,t) \Rightarrow \frac{\partial D}{\partial t}(\xi,t) = \beta E(\xi,t) \tag{6.80}$$

Differentiating (6.78) with respect to ξ as well, we get

$$\frac{\partial^2 W}{\partial \xi \partial t}(\xi,t) = \alpha W(\xi,t)\frac{\partial W}{\partial \xi}(\xi,t) - \alpha N(t)\frac{\partial W}{\partial \xi}(\xi,t) + \frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big)\frac{\partial Z}{\partial \xi}(\xi,t)$$

which is equivalent to the following

$$\frac{\partial E}{\partial t}(\xi,t) = \alpha\big(W(\xi,t) - N(t)\big)E(\xi,t) + \frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big)D(\xi,t)$$

Assume that $\beta > 0$. Denote by $P(\xi,t) := \frac{\partial Z}{\partial t}(\xi,t)$.
Using (6.77) then $W(\xi,t) - N(t) = \frac{1}{\beta}P(\xi,t)$ and we obtain

$$\frac{\partial E}{\partial t}(\xi,t) = \frac{\alpha}{\beta}P(\xi,t)E(\xi,t) + D(\xi,t)\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big) \tag{6.81}$$

Therefore, gathering all the differential equations obtained above we get the following system of ODE's:

$$\frac{\partial Z}{\partial t}(\xi,t) = \beta(W(\xi,t) - N(t)), \qquad Z(\xi,0) = \xi > 0, \tag{6.82}$$

$$\frac{\partial W}{\partial t}(\xi,t) = \frac{\alpha}{2}W^2(\xi,t) - \alpha N(t)W(\xi,t) + F\big(Z(\xi,t),t\big), \quad W(\xi,0) = h(\xi) \geq 0, \tag{6.83}$$

$$\frac{dN(t)}{dt} = -\frac{\alpha}{2}N^2(t) + Q(t), \qquad N(0) = h(0) = N_0 \tag{6.84}$$

$$\frac{\partial D}{\partial t}(\xi,t) = \beta E(\xi,t), \qquad D(\xi,0) = \frac{\partial Z}{\partial \xi}(\xi,0) = 1 \tag{6.85}$$

$$\frac{\partial E}{\partial t}(\xi,t) = \frac{\alpha}{\beta}P(\xi,t)E(\xi,t) + D(\xi,t)\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big), \qquad E(\xi,0) = \frac{\partial W}{\partial \xi}(\xi,0) = h'(\xi) \tag{6.86}$$

and $P(\xi,0) = \frac{\partial Z}{\partial t}(\xi,0) = \beta\big(W(\xi,0) - N(0)\big) = \beta\big(h(\xi) - h(0)\big)$.

If we take the equations (6.83) and (6.84) and subtract them, then we get

$$\frac{\partial W}{\partial t}(\xi,t) - \frac{dN(t)}{dt} = \frac{\alpha}{2}\big(W(\xi,t) - N(t)\big)^2 + F\big(Z(\xi,t),t\big) - Q(t) \tag{6.87}$$

or denoting by $F\big(Z(\xi,t),t\big) - Q(t) = R\big(Z(\xi,t),t\big)$ and taking into account (6.82) (by differentiating it w.r.t. t) we get

$$\frac{\partial W}{\partial t}(\xi,t) - \frac{dN(t)}{dt} = \frac{1}{\beta}\frac{\partial^2 Z}{\partial t^2}(\xi,t) = \frac{1}{\beta}\frac{\partial P}{\partial t}(\xi,t)$$

and (6.87) becomes

$$\frac{\partial P}{\partial t}(\xi,t) = \frac{\alpha}{2\beta}P^2(\xi,t) + \beta R\big(Z(\xi,t),t\big) \text{ with initial condition}$$
$$P(\xi,0) = \beta\big(h(\xi) - h(0)\big) < 0 \text{ (since } h \text{ is decreasing } \forall \xi > 0) \tag{6.88}$$

By differentiating (6.85) w.r.t. t we get $\frac{\partial^2 D}{\partial t^2}(\xi,t) = \beta\frac{\partial E}{\partial t}(\xi,t)$ and by (6.86) we get

$$\frac{\partial^2 D}{\partial t^2}(\xi,t) - \frac{\alpha}{\beta}P(\xi,t)\frac{\partial D}{\partial t}(\xi,t) - \beta D(\xi,t)\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big) = 0$$
$$D(\xi,0) = 1$$
$$\frac{\partial D}{\partial t}(\xi,0) = \beta E(\xi,0) = h'(\xi) \text{ (the initial conditions)}$$

Therefore the above system of ODE's: (6.82)-(6.86) can be rewritten in the following form

$$\frac{\partial^2 D}{\partial t^2}(\xi,t) - \frac{\alpha}{\beta}P(\xi,t)\frac{\partial D}{\partial t}(\xi,t) - \beta\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big)D(\xi,t) = 0 \tag{6.89}$$
$$\text{with initial conditions: } D(\xi,0) = 1;\ \frac{\partial D}{\partial t}(\xi,0) = \beta h'(\xi)$$
$$\frac{\partial P}{\partial t}(\xi,t) = \frac{\alpha}{2\beta}P^2(\xi,t) + \beta R\big(Z(\xi,t),t\big) \tag{6.90}$$
$$\text{with initial condition: } P(\xi,0) = \beta\big(h(\xi) - h(0)\big)$$

Next let us consider the partial differential equation (6.62) derived from the pure coagulation equation (6.52) subject to the initial condition (6.53).

Proceeding in the same way as we did with the system of ODE's (6.82)-(6.86) above, we obtain:

$$\frac{\partial Z}{\partial t}(\xi,t) = \beta(W(\xi,t) - N(t)), \qquad Z(\xi,0) = \xi > 0, \tag{6.91}$$
$$\frac{\partial W}{\partial t}(\xi,t) = \frac{\alpha}{2}W^2(\xi,t) - \alpha N(t)W(\xi,t), \qquad W(\xi,0) = h(\xi) \geq 0, \tag{6.92}$$
$$\frac{dN(t)}{dt} = -\frac{\alpha}{2}N^2(t), \qquad N(0) = h(0) = N_0 \geq 0 \tag{6.93}$$
$$\frac{\partial D}{\partial t}(\xi,t) = \beta E(\xi,t), \qquad D(\xi,0) = 1 \tag{6.94}$$
$$\frac{\partial E}{\partial t}(\xi,t) = \frac{\alpha}{\beta}P(\xi,t)E(\xi,t), \qquad E(\xi,0) = h'(\xi) \leq 0 \tag{6.95}$$

and as we have previously proved, the new system (6.91)-(6.95) can be rewritten as well as a system of two ordinary differential equations

$$\frac{\partial^2 D}{\partial t^2}(\xi,t) - \frac{\alpha}{\beta}P(\xi,t)\frac{\partial D}{\partial t}(\xi,t) = 0 \tag{6.96}$$

with initial conditions: $D(\xi,0) = 1$; $\dfrac{\partial D}{\partial t}(\xi,0) = \beta h'(\xi)$

$$\frac{\partial P}{\partial t}(\xi,t) = \frac{\alpha}{2\beta}P^2(\xi,t) \tag{6.97}$$

with initial condition: $P(\xi,0) = \beta\big(h(\xi) - h(0)\big)$

In the case of the coagulation equation with non-negative source term, the system of equations (6.82)-(6.86) can no longer be solved exactly. Our main purpose is to obtain a lower bound on the gelation time T_{gel}.

Theorem 6.8. If: $T \in (0,\infty)$ is some fixed positive constant
(i) $N \in L^1_+(0,R)$, for all $R > 0$;
(ii) α and β are non-negative constants with $\beta \neq 0$;
(iii) h is completely monotonic function for $z \geq 0$ and satisfies $h(0) = N_0 := N(0)$;
(iv) the initial condition $c_0(x) \geq 0$, $\forall x \geq 0$ and $\theta(x)c_0(x) \in L^1(0,\infty)$
(v) the source term $g \in L^1(0,T;X^+)$ for $T \in (0,\infty)$, $g(x,t) \neq 0$, for any $x \in (0,\infty)$ and all $t \in [0,T)$
(vi) the old gelation time is given by $T_0 = \frac{-1}{\beta h'(\xi)}$;
(vii) $F(z,t) = \mathcal{L}\{g(x,t)\theta(x)\}(z,t) = \int_0^\infty e^{-xz}g(x,t)\theta(x)\,dx < \infty$
Let T_1 be the first positive root of (6.89). Then the following hold

(a)

$$T_1 \leq T_{gel} := \inf\{t > 0 :\ M_1(t) < M_1(0)\} \tag{6.98}$$

(b)

$$T_1 \leq T_0 := \inf\left\{t > 0 : \text{ such that } D(\xi,t) := \frac{\partial Z}{\partial \xi}(\xi,t) = 0\right\} \tag{6.99}$$

where $D(\xi,t)$ satisfies (6.96).

Proof. (a) For $0 < t < T_1$, the method of proof of [39] shows that the coagulation equation (6.52) has a mass-conserving solution. This means that $T_1 \leq T_{gel}$ and (6.98) is proved.
(b) Let $T > 0$ be arbitrary defined by $T = \min\{T_0, T_1\}$. Let $D_1(\xi,t)$ be a solution of the equation (6.89). We have

$$\frac{\partial^2 D_1}{\partial t^2}(\xi,t) - \frac{\alpha}{\beta}P(\xi,t)\frac{\partial D_1}{\partial t}(\xi,t) - \beta\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big)D_1(\xi,t) = 0 \tag{6.100}$$

We will make use of the following transformation

$$D_1(\xi,t) = S_1(\xi,t)e^{\frac{1}{2}\int_0^t(\frac{\alpha}{\beta}P(\xi,s))\,ds} \tag{6.101}$$

Calculating the partial derivatives $\frac{\partial D_1}{\partial t}(\xi,t)$ and $\frac{\partial^2 D_1}{\partial t^2}(\xi,t)$, substituting their values in the equation (6.100) and dividing the resulted equation by $e^{\frac{1}{2}\int_0^t(\frac{\alpha}{\beta}P(\xi,s))\,ds}$, (6.100) becomes

$$\frac{\partial^2 S_1}{\partial t^2}(\xi,t) + \Big[\frac{\alpha}{2\beta}\frac{\partial P}{\partial t}(\xi,t) + (\frac{\alpha}{2\beta})^2 P^2(\xi,t) - \beta\frac{\partial F}{\partial Z}(Z(\xi,t),t)\Big]S_1(\xi,t) = 0 \tag{6.102}$$

Since $D_1(\xi,0) = 1$ and $P(\xi,0) = \beta\big(h(\xi) - h(0)\big)$ then the initial conditions of the above equation are

$$\begin{aligned} S_1(\xi,0) &= 1, \\ \frac{\partial S_1}{\partial t}(\xi,0) &= \beta\big[h'(\xi) - \frac{\alpha}{2}\big(h(\xi) - h(0)\big)\big] \end{aligned}$$

Dividing (6.102) by $S_1(\xi,t)$ we get

$$-\frac{1}{S_1(\xi,t)}\frac{\partial^2 S_1}{\partial t^2}(\xi,t) = \frac{\alpha}{2\beta}\frac{\partial P}{\partial t}(\xi,t) + \Big(\frac{\alpha}{2\beta}\Big)^2 P^2(\xi,t) - \beta\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big) \tag{6.103}$$

Take now $u_1(\xi,t) = -\frac{1}{S_1(\xi,t)}\frac{\partial S_1}{\partial t}(\xi,t)$. Differentiating u_1 w.r.t t we get

$$\frac{\partial u_1}{\partial t}(\xi,t) = -\frac{\partial^2 S_1}{\partial t^2}(\xi,t)\frac{1}{S_1(\xi,t)} + u_1^2(\xi,t)$$

Plug in $-\frac{1}{S_1(\xi,t)}\frac{\partial^2 S_1}{\partial t^2}(\xi,t) = \frac{\partial u_1}{\partial t}(\xi,t) - u_1^2(\xi,t)$ in equation (6.103) we obtain

$$\frac{\partial u_1}{\partial t}(\xi,t) = u_1^2(\xi,t) + \frac{\alpha}{2\beta}\frac{\partial P}{\partial t}(\xi,t) + \Big(\frac{\alpha}{2\beta}\Big)^2 P^2(\xi,t) - \beta\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big) \tag{6.104}$$

Therefore, using transformation (6.101), equation (6.100) becomes the Ricatti equation (6.104) above.

Let $D_2(\xi,t)$ be a solution of the equation (6.96). We have:

$$\frac{\partial^2 D_2}{\partial t^2}(\xi,t) - \frac{\alpha}{\beta}P(\xi,t)\frac{\partial D_2}{\partial t}(\xi,t) = 0 \tag{6.105}$$

Applying a similar transformation: $D_2(\xi,t) = S_2(\xi,t)e^{\frac{1}{2}\int_0^t\frac{\alpha}{\beta}P(\xi,s)\,ds}$ to the equation (6.105), calculating the partial derivatives $\frac{\partial D_2}{\partial t}(\xi,t)$ and $\frac{\partial^2 D_2}{\partial t^2}(\xi,t)$, substituting their

values in the equation (6.105) and dividing the resulted equation by $e^{\frac{1}{2}\int_0^t(\frac{\alpha}{\beta}P(\xi,s))\,ds}$, equation (6.105) becomes

$$\frac{\partial^2 S_2}{\partial t^2}(\xi,t) + \Big[\frac{\alpha}{2\beta}\frac{\partial P}{\partial t}(\xi,t) + (\frac{\alpha}{2\beta})^2 P^2(\xi,t)\Big] S_2(\xi,t) = 0 \tag{6.106}$$

with initial conditions: $S_2(\xi,0) = 1$; $\frac{\partial S_2}{\partial t}(\xi,0) = \beta\big[h'(\xi) - \frac{\alpha}{2}\big(h(\xi) - h(0)\big)\big]$. Dividing (6.106) by $S_2(\xi,t)$ we get

$$-\frac{1}{S_2(\xi,t)}\frac{\partial^2 S_2}{\partial t^2}(\xi,t) = \frac{\alpha}{2\beta}\frac{\partial P}{\partial t}(\xi,t) + \Big(\frac{\alpha}{2\beta}\Big)^2 P^2(\xi,t) \tag{6.107}$$

Taking $u_2(\xi,t) = -\frac{1}{S_2(\xi,t)}\frac{\partial S_2}{\partial t}(\xi,t)$. Differentiating u_2 with respect to t we get

$$\frac{\partial u_2}{\partial t}(\xi,t) = -\frac{\partial^2 S_2}{\partial t^2}(\xi,t)\frac{1}{S_2(\xi,t)} + u_2^2(\xi,t)$$

Plug in $-\frac{1}{S_2(\xi,t)}\frac{\partial^2 S_2}{\partial t^2}(\xi,t) = \frac{\partial u_2}{\partial t}(\xi,t) - u_2^2(\xi,t)$ in equation (6.107) we obtain the following Ricatti equation

$$\frac{\partial u_2}{\partial t}(\xi,t) = u_2^2(\xi,t) + \frac{\alpha}{2\beta}\frac{\partial P}{\partial t}(\xi,t) + \Big(\frac{\alpha}{2\beta}\Big)^2 P^2(\xi,t) \tag{6.108}$$

According to the Theorem 2.10, Chapter 2, equations (6.104), (6.108) have unique solutions since the function

$$f(u(\xi,t),t) := u^2(\xi,t) + \frac{\alpha}{2\beta}\frac{\partial P}{\partial t}(\xi,t) + \Big(\frac{\alpha}{2\beta}\Big)^2 P^2(\xi,t)$$

is continuous on $(0,\infty)\times[0,T)$ and also it is globally Lipschitz continuous with respect to u, in the sense of Definition 2.13, Chapter 2. Let u_1 and u_2 be the unique solutions of equations (6.104) and (6.108), respectively. The Ricatti equations they satisfty can be expressed in terms of function f as follows:

$$\frac{\partial u_1}{\partial t}(\xi,t) = f(u_1(\xi,t),t) - \beta\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big) \tag{6.109}$$

$$\frac{\partial u_2}{\partial t}(\xi,t) = f(u_2(\xi,t),t) \tag{6.110}$$

and u_1 and u_2 satisfy the same initial conditions: $u_1(\xi,0) = u_2(\xi,0) = -\beta h'(\xi) + \frac{\alpha}{2}\big(h(\xi) - h(0)\big)$.

Since the source term $g(x,t) \geq 0$ then $\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big) = -\int_0^\infty ze^{-zx}g(x,t)\theta(x)\,dx \leq 0$ and thus $-\beta\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big) \geq 0$ in the equation (6.109). Then we get

$$\begin{cases} \frac{\partial u_1}{\partial t}(\xi,t) & \geq f(u_1,t), \text{with initial condition} \\ u_1(\xi,0) & = -\beta h'(\xi) + \frac{\alpha}{2}\big(h(\xi) - h(0)\big) \end{cases} \text{ and}$$

$$\begin{cases} \frac{\partial u_2}{\partial t}(\xi,t) & = f(u_2,t), \text{with initial condition} \\ u_2(\xi,0) & = -\beta h'(\xi) + \frac{\alpha}{2}\big(h(\xi) - h(0)\big) \end{cases}$$

Since $u_1(\xi,0) = u_2(\xi,0)$ and f is a continuous function on an open connected set $\Omega \subset D$, where D is the domain where f is defined, then applying the inequality Lemma 2.2, Chapter 2 (see [24], Hale, p.31 and [36]) it will result that $u_1(\xi,t) \geq u_2(\xi,t)$. Since $u_1(\xi,t) = -\frac{1}{S_1(\xi,t)}\frac{\partial S_1}{\partial t}(\xi,t)$ and $u_2(\xi,t) = -\frac{1}{S_2(\xi,t)}\frac{\partial S_2}{\partial t}(\xi,t)$ then we have

$$\frac{1}{S_1(\xi,t)}\frac{\partial S_1}{\partial t}(\xi,t) \leq \frac{1}{S_2(\xi,t)}\frac{\partial S_2}{\partial t}(\xi,t)$$

Integrating both sides over $[0,t]$ and using the initial conditions $S_1(\xi,0) = S_2(\xi,0) = 1$ we get

$$\ln|S_1(\xi,t)| - \ln|S_1(\xi,0)| \leq \ln|S_2(\xi,t)| - \ln|S_2(\xi,0)|$$

or $|S_1(\xi,t)| \leq |S_2(\xi,t)|$. Multiplying both sides of the above inequality by $e^{\frac{\alpha}{2\beta}\int_0^t P(\xi,s)\,ds}$ we get

$$|D_1(\xi,t)| \leq |D_2(\xi,t)|, \forall t \in [0,T)$$

At $t = 0 : D_1(\xi,0) = D_2(\xi,0) = 1$. We are interested only in the case when $D_1(\xi,t) > 0$, for any $t \in [0,T)$. It was proved in [39] that the solution $D_2(\xi,t)$ is an increasing function of ξ and thus for any $t \in [0,T)$ we have $D_2(\xi,t) > D_2(\xi,0) > 0$, since $D_2(\xi,0) = 1$.
This implies $D_1(\xi,t) \leq D_2(\xi,t), \forall t \in [0,T)$. Since $\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big) \leq 0$, F is a non-increasing function with respect to Z then we get

$$F\big(Z(\xi,t),t\big) \leq F(0,t) = Q(t) \Rightarrow R\big(Z(\xi,t),t\big) \leq 0$$

Let's return to the equations (6.100), (6.105). Let us transform them into linear equations of the general form

$$\begin{cases} (L_1^*) & \frac{\partial}{\partial t}\big(K_1(\xi,t)\frac{\partial y}{\partial t}(\xi,t)\big) + G_1(\xi,t)y(\xi,t) = 0 \\ (L_2^*) & \frac{\partial}{\partial t}\big(K_2(\xi,t)\frac{\partial y}{\partial t}(\xi,t)\big) + G_2(\xi,t)y(\xi,t) = 0 \end{cases}$$

In order to do this we multiply both equations (6.100) and (6.105) by $e^{-\frac{\alpha}{\beta}\int_0^t P(\xi,s)\,ds}$ and we get

$$\begin{cases} (L_1) & \frac{\partial}{\partial t}\Big(\frac{\partial D_1}{\partial t}(\xi,t)e^{-\frac{\alpha}{\beta}\int_0^t P(\xi,s)\,ds}\Big) + \Big(-\beta e^{-\frac{\alpha}{\beta}\int_0^t P(\xi,s)\,ds}\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big)\Big)D_1(\xi,t) = 0 \\ (L_2) & \frac{\partial}{\partial t}\big(\frac{\partial D_2}{\partial t}(\xi,t)e^{-\frac{\alpha}{\beta}\int_0^t P(\xi,s)\,ds}\big) + 0 \cdot D_2(\xi,t) = 0 \end{cases}$$

Let us denote by

$$\begin{cases} K_1(\xi,t) & := e^{-\frac{\alpha}{\beta}\int_0^t P(\xi,s)\,ds} \\ G_1(\xi,t) & := -\beta e^{-\frac{\alpha}{\beta}\int_0^t P(\xi,s)\,ds}\frac{\partial F}{\partial Z}\big(Z(\xi,t),t\big) \end{cases}$$

and

$$\begin{cases} K_2(\xi,t) & := e^{-\frac{\alpha}{\beta}\int_0^t P(\xi,s)\,ds} = K_1(\xi,t) \\ G_2(\xi,t) & = 0 \end{cases}$$

$K_1(\xi,t)$, $K_2(\xi,t)$, $G_1(\xi,t)$, $G_2(\xi,t)$ are assumed to be continuous functions defined for any $\xi > 0$ and $t \in [0,T)$. Moreover, the functions defined above satisfy the following inequalities

$$K_2(\xi,t) \geq K_1(\xi,t) > 0 \text{ and}$$
$$0 = G_2(\xi,t) \leq G_1(\xi,t)$$

This means that the linear equation (L_1) is a Sturm majorant for the linear equation (L_2) in the sense of Definition 2.14 in Chapter 2. Since $D_1(\xi,t)$ and $D_2(\xi,t)$ are solutions of the linear equations (L_1) and (L_2), both continuous functions together with their first derivatives throughout the interval $[0,T)$, where T was defined as $T = \min\{T_0, T_1\}$, then according to the Sturm's First Comparison Theorem 2.12 in Chapter 2 (see also [27], p.334, [36]) it will result that $T_1 \leq T_0$.

In order to prove this, we use a contradiction argument. Indeed, let us assume for a contradiction that $T_0 < T_1$ (**) holds. With this assumption we have $T = T_0$. As was proved in [39], the solution $D_2(\xi,t)$ of (6.96) has exactly one zero (or root) on $[0,T_0)$. Let $D_1(\xi,t)$ be a nonzero solution of (6.89). It is clear that D_1 and D_2 satisfy (2.13), since $D_1(\xi,0) = D_2(\xi,0) = 1$, $D_1'(\xi,0) = D_2'(\xi,0) = \beta h'(\xi)$, $K_1(\xi,0) = K_2(\xi,0) = 1$. Then according to the Sturm's First Comparison Theorem 2.12 it will result that $D_1(\xi,t)$ has at least $n \geq 1$ zeros on $[0,T_0)$.

Since we have already proved that $D_1(\xi,t) \leq D_2(\xi,t)$, for any $t \in [0,T_0)$ then it follows that $D_1(\xi,t)$ has its first or smallest root, which was defined as T_1, in the interval $[0,T_0)$. In other words, we have $T_1 \leq T_0$, contradicting therefore our assumption (**). Therefore, $T_1 \leq T_0$ and Theorem 6.8 is now proved. □

Remark Since $G_2(\xi,t) \leq G_1(\xi,t)$ then according to the first Sturm's comparison Theorem 2.12) it results that $D_1(\xi,t)$ has at least as many zeros as $D_2(\xi,t)$ on $[0,T_0)$. If we denote by N_* the number of zeros of (L_1) and by N^* the number of zeros of (L_2) then we have $1 = N^* \leq N_*$. Therefore $N_* \geq 1$.

Therefore, in case the coagulation equation has a non-negative source term two conclusions hold:
(i) The lower bound T_1 appears sooner than the old gelation time T_0. This supports the hypothesis that T_0 itself appears earlier than the sourceless gelation time T_{gel}, or $T_0 \leq T_{gel}$. In fact, in [39] it was proved that $T_0 = T_{gel}$.
(ii) The number of zeros of the coagulation equation $N_* \geq 1$ (which means its solution has roots which oscillates on $[0,T_0]$).

Taking into account Theorem 6.7 we may imply the following important result regarding the existence of a mass-conserving solution of the coagulation equation (6.67):

Corollary 6.2. Let $C_0(z) = \mathcal{L}[c_0(x)](z)$. If $c_0 \in X^+$ and if $C_0''(0^+) < \infty$ then the coagulation equation (6.67) with the initial condition (6.68) has a unique mass-conserving solution $c(x,t)$, for all $t \in [0, T_1)$ for some $T_1 < 1/(\beta(\alpha + \beta C_0''(0^+))$, for any $\beta > 0$, where $C_0''(0^+) = \int_0^\infty x^2 c_0(x)\, dx$.

Conclusion.

In this chapter we presented some recent results regarding the mass-conserving solutions and the occurence of gelation for some type of coefficient kernels in the pure coagulation equations. In Section 6.1 we have also presented one of the important results given by Laurençot with respect to the large-time behaviour of the total mass for the particular case of a bilinear kernel (Theorem 6.1). This theorem provided us with an upper bound on the sourceless gelation time.

Theoretical results regarding the long-term behaviour of the total mass were investigated in the case of the coagulation equations with source term. In the second section, we obtained an upper bound for the gelation time (Theorem 6.4) for the coagulation equation with source term.

We also proved that the temporal decay of the total mass depends strongly on the amount of clusters of very small size ($x \approx 0$) in the initial distribution and in the source terms g and the smaller these quantities are, the faster is the decay of the total mass M_1, (Theorem 6.5).

Along with the results presented in Theorem 6.1, in Theorem 6.8 we provided an upper bound and a lower bound for the old gelation time.

Chapter 7

On the Formal solutions of the coagulation equation with sources

In this chapter of the thesis we are concerned with the study of solutions of the coagulation equation with source terms:

$$\frac{\partial c}{\partial t}(x,t) = \frac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)\,dy - c(x,t)\int_0^\infty K(x,y)c(y,t)\,dy + g(x,t), \tag{7.1}$$

subject to initial condition

$$c(x,0) = c_0(x) \geq 0 \qquad \text{a.e.} \tag{7.2}$$

Solvability, existence of the time-dependent problem (7.1), (7.2) for coagulation kernels of the product type have been studied in Chapter 5. The method that we use in order to carry out our analysis in the first part (Case(I) below) is the method of Laplace transforms, approach first used in [18] and continued then in [39]. In the method that we apply, there are no convergence arguments. Questions regarding the convergence have to be answered. For practical applications, either an analytical proof of convergence, or a numerical method of computing the inverse Laplace transform has to be used.

Therefore, our main objective in this chapter is to reveal some formal series solutions of the coagulation equation with constant kernels and two special cases of sources:
(I) Source terms independent of time t;
(II) Source terms possessing arbitrary time dependence;

In this chapter, we restrict our attention to the case: $K(x,y)$ symmetric, non-negative kernel given by $K(x,y) = constant = \beta \in \mathbb{R}_+$

CASE (I) Let $g(x,t) = g(x)$ be a source term independent of time. The case $K \equiv \beta$ can be transformed into the case $K \equiv 1$ by using the following change of

variables: $\tau = \beta t$. Dividing the equation (7.1) by $\beta \neq 0$ and denoting by: $d(x,\tau) := c(x,t)$ and $f(x) := \frac{1}{\beta} g(x)$ equation (7.1) becomes

$$\frac{\partial d}{\partial \tau}(x,\tau) = \frac{1}{2}\int_0^x d(x-y,\tau)d(y,\tau)\,dy - d(x,\tau)\int_0^\infty d(y,\tau)\,dy + f(x), \tag{7.3}$$

$$d(x,0) = d_0(x) \geq 0 \tag{7.4}$$

The convolution product form in the right-hand side of equation (7.3) suggests the use of the Laplace transforms. Therefore, let us denote by:

$$Y(z,\tau) := \mathcal{L}\{d(x,\tau)\}(z,\tau) = \int_0^\infty e^{-zx} d(x,\tau)\,dx$$

$$G(z) := \mathcal{L}\{f(x)\}(z) = \int_0^\infty e^{-zx} f(x)\,dx$$

$$h(z) := \mathcal{L}\{d_0(x)\}(z) = \int_0^\infty e^{-zx} d_0(x)\,dx$$

Take formally Laplace transform of (7.3) and use the property:
$\mathcal{L}\{(d*d)(x,\tau)\}(z,\tau) = \mathcal{L}\{d(x,\tau)\}^2(z,\tau)$, where "$*$" represents the convolution product $(d*d)(x,\tau) := \int_0^x d(x-y,\tau)d(y,\tau)\,dy$. We have

$$\frac{\partial Y(z,\tau)}{\partial \tau} = \frac{1}{2}Y^2(z,\tau) - M_0(\tau)Y(z,\tau) + G(z) \tag{7.5}$$

where $M_0(\tau) := \int_0^\infty d(x,\tau)\,dx$ is the zero moment of the solution d of equation (7.3). At $z = 0$ we have $Y(0,\tau) = \int_0^\infty d(x,\tau)\,dx = M_0(\tau)$ and equation (7.5) takes the form:

$$\frac{dM_0(\tau)}{d\tau} = -\frac{1}{2}M_0^2(\tau) + G(0) \tag{7.6}$$

$$M_0(0) = \int_0^\infty d_0(x)\,dx =: h(0) \qquad \text{given} \tag{7.7}$$

Equation (7.6) is a Riccati equation which is readily solved. The solution of the initial value problem (7.6), (7.7) is given by:

$$M_0(\tau) = \sqrt{2G(0)}\tanh\left\{\sqrt{\frac{G(0)}{2}}\tau + \tanh^{-1}\left(\frac{h(0)}{\sqrt{2G(0)}}\right)\right\} \tag{7.8}$$

Moreover, gathering both I.V.P's we have

$$\frac{dM_0(\tau)}{d\tau} = -\frac{1}{2}M_0^2(\tau) + G(0), \qquad M_0(0) = h(0) \geq 0, \tag{7.9}$$

$$\frac{\partial Y(z,\tau)}{\partial \tau} = \frac{1}{2}Y^2(z,\tau) - M_0(\tau)Y(z,\tau) + G(z), \qquad Y(z,0) = h(z) \geq 0, \tag{7.10}$$

Subtracting (7.9), (7.10) and denoting by $R(z,\tau) := M_0(\tau) - Y(z,\tau)$ we get the following nonlinear first order equation (of Riccati type):

$$\frac{\partial R(z,\tau)}{\partial \tau} = -\frac{1}{2}R^2(z,\tau) + G(0) - G(z)$$
$$R(z,0) = h(0) - h(z)$$

Denote by $H(z) := h(0) - h(z) = \int_0^\infty c_0(x)(1-e^{-xz})\,dx \geq 0$ and $V(z) := G(0) - G(z) = \int_0^\infty f(x)(1-e^{-xz})\,dx \geq 0$. We have

$$\frac{\partial R(z,\tau)}{\partial \tau} = -\frac{1}{2}R^2(z,\tau) + V(z) \tag{7.11}$$
$$R(z,0) = H(z) \geq 0 \tag{7.12}$$

The solution of the initial value problem (7.11), (7.12) is

$$R(z,\tau) = \sqrt{2V(z)}\tanh\left\{\sqrt{\frac{V(z)}{2}}\tau + \tanh^{-1}\left[\frac{H(z)}{\sqrt{2V(z)}}\right]\right\} \tag{7.13}$$

Therefore, using the solutions (7.8), (7.13) obtained above we get

$$Y(z,\tau) = M_0(\tau) - R(z,\tau) = \sqrt{2G(0)}\tanh\left\{\sqrt{\frac{G(0)}{2}}\tau + \tanh^{-1}\left[\frac{h(0)}{\sqrt{2G(0)}}\right]\right\}$$
$$- \sqrt{2V(z)}\tanh\left\{\sqrt{\frac{V(z)}{2}}\tau + \tanh^{-1}\left(\frac{H(z)}{\sqrt{2V(z)}}\right)\right\} \tag{7.14}$$

Since $d(x,\tau) = \mathcal{L}^{-1}\{Y(z,\tau)\}(x,\tau) = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} Y(z,\tau)e^{xz}\,dz$, then in order to find the solution $d(x,\tau)$, we need to find the inverse Laplace transform of $M_0(\tau)$ and $R(z,\tau)$, respectively. We have $\mathcal{L}^{-1}\{M_0(\tau)\}(x,\tau) = M_0(\tau)\delta(x)$. In order to compute the inverse Laplace transform of $R(z,\tau)$ we either express $R(z,\tau)$ as a series expansion or we make use of numerical methods. Since in most cases the series are not likely to be convergent, the second method is more reliable. Let us begin with the simplest case.

Case(I): Example Let us assume first that no particles were present before $\tau = 0$, so $c_0(x) = d_0(x) = 0$. From this we get $h(z) = h(0) = 0$ and $H(z) = 0$ and problem (7.11), (7.12) becomes

$$\frac{\partial R(z,\tau)}{\partial \tau} = -\frac{1}{2}R^2(z,\tau) + V(z) \tag{7.15}$$
$$R(z,0) = 0 \tag{7.16}$$

The solution of this IVP is given by

$$R(z,\tau) = \sqrt{2V(z)}\tanh\left\{\sqrt{\frac{V(z)}{2}}\tau\right\}.$$

Thus the solution $Y(z,\tau)$ of equation (7.10) becomes

$$Y(z,\tau) = \sqrt{2G(0)}\tanh\left[\sqrt{\frac{G(0)}{2}}\tau\right] - \sqrt{2V(z)}\tanh\left[\sqrt{\frac{V(z)}{2}}\tau\right]$$

Using the Taylor series expansion for tanh, i.e. $\tanh(z) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} z^{2n-1}$ which is valid for $|z| < \pi/2$ (where the B_{2n} are the Bernoulli numbers) we get,

$$\begin{aligned} R(z,\tau) &= \sqrt{2V(z)} \cdot \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} \sqrt{V(z)}^{2n-1} \left(\frac{\tau}{\sqrt{2}}\right)^{2n-1} \\ &= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} \frac{\tau^{2n-1}}{2^{n-1}} B_{2n} V(z)^n \end{aligned}$$

Using the linearity properties of the inverse Laplace transform we have

$$\mathcal{L}^{-1}\{R(z,\tau)\}(x,\tau) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} \frac{\tau^{2n-1}}{2^{n-1}} B_{2n} \mathcal{L}^{-1}\{V(z)^n\}. \tag{7.17}$$

In order to give a complete solution, we present here some particular examples of source terms, $f(x)$, independent of time τ.

Example 7.1. Let $f(x) = \delta(x-1)$.

In this case, we have $V(z) = \int_0^{\infty} \delta(x-1)(1-e^{-xz})\,dx = 1 - e^{-z}$. Now, using the Binomial Theorem (since $|e^{-z}| < 1$, for all z), we have

$$\begin{aligned} (1-e^{-z})^n &= \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)(-1)^k}{k!} e^{-kz} \qquad \Rightarrow \\ \mathcal{L}^{-1}\{(1-e^{-z})^n\}(x) &= \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)(-1)^k}{k!} \delta(x-k) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}^{-1}\{R(z,\tau)\}(x,\tau) = \sum_{n=1}^{\infty} &\frac{2^{2n}(2^{2n}-1)}{(2n)!} \frac{\tau^{2n-1}}{2^{n-1}} B_{2n} \cdot \\ &\cdot \left[\sum_{k=0}^{\infty} (-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!} \delta(x-k)\right]. \end{aligned} \tag{7.18}$$

Using $\mathcal{L}^{-1}\{M_0(\tau)\}(x,\tau) = M_0(\tau)\delta(x) = \sqrt{2}\tanh\left(\frac{\sqrt{2}}{2}\tau\right)\delta(x)$ and the above, a formal series solution of the coagulation equation (7.3), (7.4) finally takes the form:

$$d(x,\tau) = \mathcal{L}^{-1}\{Y(z,\tau)\}(x,\tau) = \sqrt{2}\tanh\left(\frac{\sqrt{2}}{2}\tau\right)\delta(x) - \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}\frac{\tau^{2n-1}}{2^{n-1}}B_{2n}\Big[\sum_{k=0}^{\infty}(-1)^k \frac{n(n-1)\cdots(n-k+1)}{k!}\delta(x-k)\Big]. \tag{7.19}$$

Example 7.2. Let $f(x) = e^{-x}$.

Here we have $V(z) = \int_0^\infty e^{-x}(1-e^{-xz})\,dx = 1 - \frac{1}{z+1}$. On using the Binomial Theorem (since $|\frac{1}{z+1}| < 1$, for all z), we have

$$\Big(1-\frac{1}{z+1}\Big)^n = \sum_{k=0}^{\infty}\frac{n(n-1)(n-2)\cdots(n-k+1)(-1)^k}{k!}\frac{1}{(z+1)^k} \quad \Rightarrow$$

$$\mathcal{L}^{-1}\Big\{\Big(1-\frac{1}{z+1}\Big)^n\Big\}(x) = \sum_{k=0}^{\infty}\frac{n(n-1)\cdots(n-k+1)(-1)^k}{k!}\mathcal{L}^{-1}\Big\{\frac{1}{(z+1)^k}\Big\}(x)$$

$$= \sum_{k=0}^{\infty}\frac{n(n-1)\cdots(n-k+1)(-1)^k}{k!}\frac{x^{k-1}e^{-x}}{\Gamma(k)}$$

where the gamma function is defined by $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}\,dt = (k-1)!$ with $k = 1,2,\ldots$. Similarly, as in Example 7.1, we find that a formal series solution of the coagulation equation (7.3), (7.4) is given by:

$$d(x,\tau) = \mathcal{L}^{-1}\{Y(z,\tau)\}(x,\tau) = \sqrt{2}\tanh\left(\frac{\sqrt{2}}{2}\tau\right)\delta(x) - \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}\frac{\tau^{2n-1}}{2^{n-1}}B_{2n}\Big[\sum_{k=0}^{\infty}\frac{n(n-1)\cdots(n-k+1)(-1)^k}{k!}\frac{x^{k-1}e^{-x}}{\Gamma(k)}\Big]. \tag{7.20}$$

CASE (II): Source terms possessing arbitrary time variation.

One of the more important generalizations is the introduction into the coagulation equation (7.1), (7.2) of a source term possessing arbitrary time variation. Proceeding similarly as in Case(I) with (7.1), (7.2), using the change of variables: $\tau = \beta t$ and denoting by: $d(x,\tau) = c(x,t)$ and $f(x,\tau) = \frac{1}{\beta}g(x,\tau)$, equation (7.1) becomes

$$\frac{\partial d}{\partial \tau}(x,\tau) = \frac{1}{2}\int_0^x d(x-y,\tau)d(y,\tau)\,dy - d(x,\tau)\int_0^\infty d(y,\tau)\,dy + f(x,\tau), \tag{7.21}$$

$$d(x,0) = c_0(x) \geq 0 \tag{7.22}$$

Using again the Laplace transform method and denoting by

$$Y(z,\tau) := \mathcal{L}\{d(x,\tau)\}(z,\tau) = \int_0^\infty e^{-zx} d(x,\tau)\, dx$$
$$R(z,\tau) := \mathcal{L}\{f(x,\tau)\}(z,\tau) = \int_0^\infty e^{-zx} f(x,\tau)\, dx$$
$$h(z) := \mathcal{L}\{d_0(x)\}(z) = \int_0^\infty e^{-zx} d_0(x)\, dx$$

Taking formally Laplace transform to (7.21) we obtain

$$\frac{\partial Y(z,\tau)}{\partial \tau} = \frac{1}{2} Y^2(z,\tau) - M_0(\tau) Y(z,\tau) + R(z,\tau) \tag{7.23}$$
$$Y(z,0) = h(z) \geq 0 \qquad \text{given}$$

where $M_0(\tau) = Y(0,\tau) = \int_0^\infty d(x,\tau)\, dx$. By letting $z = 0$ in (7.23) we obtain the equation for M_0 that is,

$$\frac{dM_0(\tau)}{d\tau} = -\frac{1}{2} M_0^2(\tau) + R(0,\tau) \tag{7.24}$$
$$M_0(0) = h(0) \geq 0 \qquad \text{(given)} \tag{7.25}$$

By letting $F(z,\tau) := M_0(\tau) - Y(z,\tau)$ and $V(z,\tau) = R(0,\tau) - R(z,\tau)$ then subtracting (7.23) from equation (7.24) we obtain the following nonlinear equation

$$\frac{\partial F(z,\tau)}{\partial \tau} = -\frac{1}{2} F^2(z,\tau) + V(z,\tau) \tag{7.26}$$
$$F(z,0) = h(0) - h(z) =: F_0(z) \geq 0 \tag{7.27}$$

In order to deal with the nonlinear equation (7.26) we define the function $G(z,\tau)$ by

$$G(z,\tau) = e^{\frac{1}{2}\int_0^\tau F(z,\xi)\, d\xi} \tag{7.28}$$

Differentiating G with respect to τ we have

$$F(z,\tau) = \frac{2G'(z,\tau)}{G(z,\tau)}, \tag{7.29}$$

where the dash notation represents the differentiation with respect to τ. It then readily follows from equation (7.28) that G satisfies

$$G''(z,\tau) - \frac{1}{2} V(z,\tau) G(z,\tau) = 0 \tag{7.30}$$

which being a linear equation in G is much easier to deal with than the nonlinear equation for F. Now since equation (7.30) is of second order, its general solution will be a linear combination of any two linearly independent solutions and thus will contain two arbitrary constants (functions of z in our case). However, one of these constants will cancel out when we obtain F from equation (7.29) and so if we denote two linearly independent solutions of equation (7.30) by G_1 and G_2 we deduce from equation (7.29) that the general solution for F is given by

$$F(z,\tau) = \frac{2(C_1(z)G_1'(z,\tau) + C_2(z)G_2'(z,\tau))}{C_1(z)G_1(z,\tau) + C_2(z)G_2(z,\tau)} = \frac{2(G_1'(z,\tau) + \alpha(z)G_2'(z,\tau))}{G_1(z,\tau) + \alpha(z)G_2(z,\tau)} \tag{7.31}$$

where $\alpha(z)$ is an arbitrary function of the parameter z. The fact that the general solution for F contains only a single arbitrary quantity $\alpha(z)$ is certainly expected since F satisfies the first order equation (7.26). By specifing $F_0(z)$ we determine $\alpha(z)$.

At this stage of work, it is worth proving that our general approach through equations (7.31) will yield the physically necessary result that $F(z,\tau) \geq 0$ for $(\tau > 0)$ if $F(z,0) = F_0(z) > 0$. Indeed, in order to prove this we multiply equation (7.30) by $G(z,\tau)$ and integrate with respect to τ from 0 to τ. We get

$$G'(z,\tau)G(z,\tau) = G'(z,0)G(z,0) + \frac{1}{2}\int_0^\tau V(z,\tau')\left[G(z,\tau')\right]^2 d\tau' + \int_0^\tau \left[G'(z,\tau')\right]^2 d\tau'$$

Divide this equality by $G^2(z,\tau) > 0 \Rightarrow$

$$\frac{G'(z,\tau)}{G(z,\tau)} = \frac{1}{G^2(z,\tau)} \cdot \left\{ \left[G(z,0)\right]^2 \frac{G'(z,0)}{G(z,0)} + \int_0^\tau \left\{ \left[G'(z,\tau')\right]^2 + \frac{1}{2}V(z,\tau')\left[G(z,\tau')\right]^2 \right\} d\tau' \right\} \tag{7.32}$$

From here it follows immediately, since $\frac{G'(z,0)}{G(z,0)} \geq 0$ and all the terms in (7.32) are non-negative, that $F(z,\tau) = \frac{2G'(z,\tau)}{G(z,\tau)} \geq 0$.

In the following we will use the general solution (7.31) of the problem (7.26) to obtain the information about its asymptotic long-term behaviour. First, let us point out here a basic similarity between the long term behaviour of the solution F of our present problem (7.26) and the long term behaviour of the solution E of the corresponding linear equation

$$\frac{dE(\tau)}{d\tau} = -\frac{1}{2}E(\tau) + V(\tau)$$

In the latter case, the general solution may be expressed as: $E(\tau) = E_1(\tau) +$

$\alpha E_2(\tau)$, where $E_1(\tau)$ and $E_2(\tau)$ satisfy the equations

$$\frac{dE_1(\tau)}{d\tau} = -\frac{1}{2}E_1(\tau) + V(\tau)$$
$$\frac{dE_2(\tau)}{d\tau} = -\frac{1}{2}E_2(\tau)$$

and α is an arbitrary constant. By specifying $E(0)$ we determine α and thus a unique solution $E(\tau)$.

The following important result illustrates the long term behaviour of the solution $E(\tau)$.

Lemma 7.1. (see Simons[42], p.3762) If $\lim_{\tau\to\infty}\left[E_2(\tau)/E_1(\tau)\right] = 0$ then the long term behaviour of $E(\tau)$ becomes independent of α and hence independent of the initial value $E(0)$. The term $\alpha E_2(\tau)$ is described as forming the "transient" component of E and the long-term behaviour of $E(\tau)$ becomes independent of this transient component, depending only on the driving (source) term $V(\tau)$.

Therefore, we can see that such a "transient" behaviour can occur with the solution (7.31) of the nonlinear problem (7.26). If we can choose a pair of linearly independent solutions $G_1(z,\tau)$ and $G_2(z,\tau)$ of equation (7.30) such that

$$\lim_{\tau\to\infty}\frac{G_2(z,\tau)}{G_1(z,\tau)} = \lim_{\tau\to\infty}\frac{G_2'(z,\tau)}{G_1'(z,\tau)} = 0$$

then equation (7.31) gives

$$\lim_{\tau\to\infty} F(z,\tau) = \frac{G_1'(z,\tau)}{G_1(z,\tau)} \tag{7.33}$$

which is independent of $\alpha(z)$ and therefore independent of $F(z,0)$. According to the above Lemma 7.1, it will result that the initial cluster distribution u_0 has only a transient effect on its future development and its long-term behaviour is determined entirely by the source term $V(z,\tau)$. Let us give here a particular example of source term $f(x,\tau)$ to illustrate the above theory.

Example 7.3. Let us assume that the source term in the coagulation equation (7.21) has the general form: $f(x,\tau) = \frac{u(x)}{(\tau+\tau_0)^2}$ where τ_0 is a real constant.

Then we have $R(z,\tau) = \int_0^\infty e^{-xz}\frac{u(x)}{(\tau+\tau_0)^2}\,dx$. Then

$$V(z,\tau) := R(0,\tau) - R(z,\tau) = \frac{1}{(\tau+\tau_0)^2}\int_0^\infty (1-e^{-xz})u(x)\,dx = \frac{H(z)}{(\tau+\tau_0)^2}$$

where we have put $H(z) = \int_0^\infty (1-e^{-xz})u(x)\,dx$. Therefore, equation (7.30) becomes

$$G''(z,\tau) - \frac{H(z)}{2}\frac{1}{(\tau+\tau_0)^2}G(z,\tau) = 0 \tag{7.34}$$

A pair of linearly independent solutions are:

$$G_1(z,\tau) = (\tau+\tau_0)^{\alpha_1};\ G_2(z,\tau) = (\tau+\tau_0)^{\alpha_2} \qquad \text{where}$$
$$\alpha_1 = \frac{1}{2}\Big(1+\sqrt{1+2H(z)}\Big);\ \alpha_2 = -\frac{1}{2}\Big(-1+\sqrt{1+2H(z)}\Big)$$

The general solution of F is given by

$$F(z,\tau) = \frac{2(G_1'(z,\tau)+\alpha(z)G_2'(z,\tau))}{G_1(z,\tau)+\alpha(z)G_2(z,\tau)}$$
$$F(z,\tau) = \frac{2(\alpha_1(\tau+\tau_0)^{\alpha_1-1}+\alpha(z)\alpha_2(\tau+\tau_0)^{\alpha_2-1})}{(\tau+\tau_0)^{\alpha_1}+\alpha(z)(\tau+\tau_0)^{\alpha_2}}. \tag{7.35}$$

It is clear that, as $\tau \to \infty$, the second term in both the numerator and the denominator of F can be neglected compared to the first terms. According to the remarks following Lemma 7.1, the long-term behaviour of $F(z,\tau)$ will be determined by the equation $A(z,\tau) = \lim_{\tau\to\infty} F(z,\tau) = \frac{2\alpha_1}{\tau+\tau_0}$ and thus by the equation

$$A(z,\tau) = \frac{1+\sqrt{1+2H(z)}}{\tau+\tau_0} \tag{7.36}$$

Or, the asymptotic long-term behaviour of F is given by $F(z,\tau) \sim \frac{1+\sqrt{1+2H(z)}}{\tau+\tau_0}$ as $\tau \to \infty$. Therefore F is independent of the initial condition $F_0(z)$.

Let us study now the behaviour of the zero-th moment of solution d of the coagulation equation (7.21). As we have already noticed, at $z = 0$ we obtain that $M_0(\tau) = Y(0,\tau)$ and $M_0(\tau)$ satisfies the nonlinear first order equation (of Ricatti type)

$$\frac{dM_0(\tau)}{d\tau} = -\frac{1}{2}M_0^2(\tau) + R(0,\tau)$$

where $R(0,\tau) = \int_0^\infty f(x,\tau)\,dx = \int_0^\infty \frac{u(x)}{(\tau+\tau_0)^2}\,dx$. Let us assume that $u(x)$ is chosen so that the following integral exists and is finite $\int_0^\infty u(x)\,dx =: C \in (0,\infty)$ (e.g. $u(x) = e^{-x}$ or $u(x) = \delta(x)$), where C is a constant. Then we have $R(0,\tau) = \frac{C}{(\tau+\tau_0)^2}$. Therefore $M_0(\tau)$ satisfies the first order nonlinear equation

$$\frac{dM_0(\tau)}{d\tau} = -\frac{1}{2}M_0^2(\tau) + \frac{C}{(\tau+\tau_0)^2} \tag{7.37}$$

In order to solve equation (7.37) we follow the same method of solution we presented above for F. We define the function $r(\tau) = e^{\frac{1}{2}\int_0^\tau M_0(\xi)\,d\xi}$. Differentiating r with respect to τ we get $M_0(\tau) = \frac{2r'(\tau)}{r(\tau)}$. Substituting this value into the equation (7.37) we obtain the second order linear differential equation in r

$$r''(\tau)(\tau+\tau_0)^2 - \frac{C}{2}r(\tau) = 0 \tag{7.38}$$

This equation is easier to deal with than the first order nonlinear equation (7.37) obtained above. A pair of linearly independent solutions for (7.38) would be

$$r_1(\tau) = (\tau+\tau_0)^{\gamma_1};\ r_2(\tau) = (\tau+\tau_0)^{\gamma_2} \qquad \text{where}$$
$$\gamma_1 = \frac{1}{2}\Big(1+\sqrt{1+2C}\Big);\ \gamma_2 = -\frac{1}{2}\Big(-1+\sqrt{1+2C}\Big)$$

Thus, as previously obtained, the general solution of r is given by $r(\tau) = r_1(\tau)+\lambda r_2(\tau)$ where λ is a constant that can be found from the initial value $r(0)$. The solution of the equation (7.37) is then given by:

$$M_0(\tau) = \frac{2(r'(\tau))}{r(\tau)} = \frac{2(r_1'(\tau)+\lambda r_2'(\tau))}{r_1(\tau)+\lambda r_2(\tau)}$$
$$M_0(\tau) = \frac{2(\gamma_1(\tau+\tau_0)^{\gamma_1-1}+\lambda\gamma_2(\tau+\tau_0)^{\gamma_2-1})}{(\tau+\tau_0)^{\gamma_1}+\lambda(\tau+\tau_0)^{\gamma_2}} \tag{7.39}$$

And for given $M_0(0)$ we find the constant λ in terms of r_1 and r_2 by solving the equation:

$$M_0(0) = \frac{2r_1'(0)+2\lambda r_2'(0)}{r_1(0)+r_2(0)} \tag{7.40}$$

It is clear that, as $\tau\to\infty$, the second term in both the numerator and the denominator of (7.39) can be neglected as compared with the first terms. Thus the long-term behaviour of $M_0(\tau)$ is given by $n(\tau) = 2\frac{\gamma_1}{(\tau+\tau_0)^2} = \frac{1+\sqrt{1+2C}}{\tau+\tau_0}$, which is independent of the initial distribution. The expression of $n(\tau)$ is proportional to $(\tau+\tau_0)^{-1}$. From the long-term behaviour of $M_0(\tau)$ and of $F(z,\tau)$ we may deduce the long term behaviour of

$$Y(z,\tau) := Y(0,\tau) - F(z,\tau) = \frac{\sqrt{1+2C}}{\tau+\tau_0} - \frac{\sqrt{1+2H(z)}}{\tau+\tau_0}$$

Thus the long term-behaviour of the solution d of the coagulation equation (7.21) can be found using the method of Laplace transforms and it is given by:

$$d(x,\tau) \sim \frac{1}{\tau+\tau_0}\sqrt{1+2C}\delta(x) - \frac{1}{\tau+\tau_0}p(x) \ \text{ as } \tau\to\infty \tag{7.41}$$

where $p(x) = \mathcal{L}^{-1}\big(1+2H(z)\big)^{\frac{1}{2}}(x)$. Therefore, from here we may deduce the long-term behaviour of the solution is independent of the initial condition and is itself proportional with $(\tau+\tau_0)^{-1}$. From the long-term behaviour of the solution d we may deduce the long-term behaviour of the first moment of the solution, which is defined by $M_1(\tau) := \int_0^\infty x d(x,\tau)\,dx$. Since $\int_0^\infty x\delta(x)\,dx = 0$ we may conclude that $M_1(\tau)$ is itself proportional with $(\tau+\tau_0)^{-1}$ and its long-term behaviour is given by

$$M_1(\tau) \sim \frac{1}{\tau+\tau_0}\int_0^\infty xp(x)\,dx \ \text{ as } \tau\to\infty \tag{7.42}$$

where $p(x)$ can be computed by using numerical methods for the inversion of the Laplace transform. Other examples for $u(x)$ include e^{-x}. It is worth pointing out here that the same long-term behaviour (7.36) will also hold for a more general time-variation in the source term $f(x,\tau)$. For instance, $f(x,\tau) = \frac{u(x)}{S(\tau)}$ with $S(\tau) = (\tau+\tau_0)^p$ with $p \geq 0$.

Conclusion. In this chapter we obtained some formal series solutions of the coagulation equation with constant kernels. Examples we considered include time-independent cases of source terms. Using the Laplace transforms we obtained a formal series solution. In this chapter there are no convergence arguments. Questions regarding the convergence have to be answered. For practical applications, either an analytical proof of convergence, or a numerical method of computing the inverse Laplace transform should be employed.

Furthermore, a theoretical treatment is developed for the solution of the coagulation equation with constant kernel in the presence of a time-dependent source term. In this chapter we have shown how the relevant first order differential equation (7.26) can be transformed into a linear second-order equation (7.30), which then can be used to obtain the general solution of the problem together with information about its asymptotic long-term behaviour.

Detailed conclusions are derived for a particular source term $f(x,\tau) = \frac{u(x)}{(\tau+\tau_0)^2}$. In this case, it was proved that the long-term behaviour of the solution of the nonlinear Ricatti equation (7.26) is independent of the initial conditions and entirely dependent on the source term, which means, in our case it is proportional to $(\tau+\tau_0)^{-1}$.

Chapter 8

Conclusion

In this thesis we have studied the dynamics of the coagulation equation with source and sink terms. In Chapter 3, using a well-known technique of re-scaling the time variable t, we simplified the Smoluchowski equation with sources and sinks to a general Smoluchowski equation with particle sources only. We introduced a suitable Banach space X and assumed that the initial distribution belongs to the positive cone of X and the source term is an essentially bounded function and integrable on the time interval $[0, T]$, for any $T > 0$.

Based on a contraction mapping argument, in Chapter 4, we proved the existence and uniqueness of a local solution in suitably defined Banach spaces, for kernels $K(x, y)$ that are bounded independently of the particle sizes x and y. Moreover, the solution of the coagulation equation was shown to exist and is unique in the Banach space X.

By investigating the properties of the solutions to the approximating equations, in Chapter 5 we proved the existence of solutions to the coagulation equations with source terms in the Banach space $X = \{f \in L^1(0, \infty) : \int_0^\infty (1 + \frac{\beta}{\alpha} x)|f(x)|\, dx < \infty\}$, with $\alpha > 0$, $\beta \geq 0$ positive constants for product-type coagulation kernels of the form: $K(x, y) = (\alpha + \beta x)(\alpha + \beta y)$.

Furthermore, in Chapter 6 we analyzed the existence of mass-conserving solutions to the coagulation equations and the occurrence of gelation for the class of product-type coagulation kernels mentioned above. We extended the results obtained for the pure coagulation equation to the coagulation equation with sources and obtained an upper bound on the gelation time. Our main result, Theorem 6.5, proves that the temporal decay of the total mass depends strongly on both the number of clusters of very small size ($x \approx 0$) in the initial distribution $u_0(x)$ and the number of new particles of very small size in the source term and we deduce that the smaller these quantities are, the faster is the decay of the total mass.

Furthermore, using the method given by Shirvani and van Roessel [39], for the pure coagulation equation with product-type kernels we obtained a lower bound for the old gelation time (Theorem 6.8). Along with Theorem 6.1, Theorem 6.8 gives us an upper and a lower bound for the gelation time with product-type coagulation

kernels.

Finally, in Chapter 7, based on the method of Laplace transforms, we revealed some formal series solutions of the coagulation equation with constant kernels. Examples that we provided include time-independent cases of source terms.

A theoretical treatment is developed for the solution of the coagulation equation with constant kernel in the presence of a source term possessing arbitrary time-dependence. In Chapter 7 we obtained information regarding the asymptotic long-term behaviour of the solution. We proved that the long-term behaviour of the solution is independent of the initial conditions and entirely dependent on the source term.

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